$$U_k = \sum_{j=0}^{[\frac{k}{2}]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j} s^j.$$

This can also be verified directly.

In a future paper we shall show that there are generating functions for the four recurrence relations given in this paper. These can also be used for the special cases of this section. We can use them to generate with a computer as many terms in a given recurrence relation as desired.

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ON GENERATING FUNCTIONS AND DOUBLE SERIES EXPANSIONS

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1. INTRODUCTION

Recently, Weiss *et al.* [9] gave a direct proof of a result due to Narayana [8] and Kreweras [6]:

$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} u^r v^s = \frac{1}{2}[1-u-v-(1-2(u+v)+(u-v)^2)^{1/2}]. \tag{1.1}$$

A special case of Theorem 1a of this pàper is a five-parameter generalization of (1.1):

$$\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha + 1 + gk + hp)} \binom{\alpha + gk + k + hp}{k} \binom{\beta + gck + hcp + p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\beta+1}}{(\alpha+1)} {}_2F_1\left[\begin{matrix} 1,\ 1+\beta-c-\alpha c, \\ (\alpha+1+h)/h, \end{matrix}\ -y\right], \tag{1.2}$$

where

$$u = \frac{z}{(1+z)^{g+1}(1+y)^{gc}}, \quad v = \frac{y}{(1+z)^h(1+y)^{hc+1}}.$$

See Luke [7, Sec. 6.10] for a discussion of Padé approximation for the hypergeometric function on the right-hand side of (1.2). Letting

$$g = -1,\ h = -1,\ c = 1,\ \alpha = -2, \text{ and } \beta = -2$$

in (1.2) and some manipulation will give (1.1).

Equation (1.2) also appears to be an extension of the important equation (6.1) of Gould [5], to which it reduces for $z = 0$.

An interesting simplification of (1.2) is the case $\beta = \alpha c + c - 1$, giving:

$$\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha + 1 + gk + hp)} \binom{\alpha + gk + k + hp}{k} \binom{\alpha c + c - 1 + gck + hcp + p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\alpha c + c}}{(1+\alpha)}. \tag{1.3}$$

The importance of these types of expansions is the connection with Jacobi polynomials. Now, Carlitz [1] gave the important generating function

$$\sum_{n=0}^{\infty} \frac{\alpha}{n+\alpha} P_n^{(\alpha,-1)}(x) r^n = 2^{\alpha}(1 - r + R)^{-\alpha}, \tag{1.4}$$

where

$$R = (1 - 2xr + r^2)^{1/2} \quad \text{and} \quad P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} {}_2F_1\left[\begin{matrix} -n,\ n+\alpha+\beta+1, \\ \alpha+1, \end{matrix}\ \frac{1-x}{2}\right].$$

A special case of Theorem 1a in this paper gives

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+\tau+1+an+bn+n)} P_n^{(\sigma+an,\tau+bn)}(w) \tag{1.5}$$

$$= \frac{1}{(\sigma+\tau+1)}(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma} {}_2F_1\left[\begin{matrix} 1,\ \{(1+a)(1+\tau)-\sigma b\}/(1+a+b), \\ (\sigma+\tau+a+b+2)/(a+b+1), \end{matrix}\ y\right],$$

where y and z are defined by

$$(1-w)/2 = z(1-y)/[y(1-z)],$$

$$\xi = y(1-y)^a/(1-z)^{a+b+1},$$

and

$$|\xi| < 1,\ |y| < 1,\ |z| < 1.$$

By letting $a = b = 0$ and $\sigma = -1$, (1.5) reduces to (1.4). See [3] for another generalization of (1.4), and some discussion regarding its importance.

A special case of interest occurs for $\tau = (\sigma b - a - 1)/(1 + a)$, giving:

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+\tau+1+an+bn+n)} P_n^{(\sigma+an,\tau+bn)}(w) = \frac{(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma}}{\sigma+\tau+1}. \tag{1.6}$$

Equation (1.5) is also a three-parameter extension of another equation of Carlitz [1, Eq. 8]. Letting $a = 0$ in (1.5) gives equation (1) of Cohen [4].

A special case of Theorem 1b of this paper yields the expression:

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{u^k v^p}{(\beta+1+gck+hcp)} \binom{\alpha+gk+k+hp}{k}\binom{\beta+gck+hcp+p}{p}$$

$$= \frac{(1+z)^{\alpha+1}(1+y)^{\beta+1}}{(\beta+1)} {}_2F_1\left[\begin{matrix} \dfrac{c-\beta-1+\alpha c}{c},\ 1, \\ \dfrac{\beta+1+gc}{gc}, \end{matrix}\ -z\right] \tag{1.7}$$

where

$$u = \frac{z}{(1+z)^{g+1}(1+y)^{gc}},\quad v = \frac{y}{(1+z)^k(1+y)^{hc+1}}.$$

The analogous expression for the Jacobi polynomial takes the form

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma+an+n)} P_n^{(\sigma+an,\tau+bn)}(w) \tag{1.8}$$

$$= \sigma^{-1}(1-z)^{\sigma+\tau+1}(1-y)^{-\sigma} {}_2F_1\left[\begin{matrix} 1,\ (1+\tau)(1+a)-\sigma b/(1+a), \\ (\sigma+a+1)/a, \end{matrix}\ z\right].$$

Letting $\tau = -1$, $a = b = 0$, the Carlitz formula given by our equation (1.4) presents itself. Also, letting $b = 0$ in (1.8) gives essentially a main result in [2, Eq. (1.1)]. [The variables y and z are defined in (1.5).]

The statement and proof of Theorem 1 follow in the next section.

2. STATEMENT AND PROOF OF THEOREM 1

Theorem 1: For a, b, c, α, and β complex numbers and ℓ, ℓ', and j nonnegative integers:

a.
$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\xi_1^k\xi_2^p\binom{\alpha+ak+k+bp}{k}\binom{\beta+ack+bcp+p}{p}}{(\alpha+ak+bp+\ell+1)\binom{(\alpha+ak+bp+\ell+1+j\ell')/j}{\ell'}}$$
$$= (1-z)^{\alpha+\ell+1}(1-y)^{\beta+1} \times$$
$$\sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\frac{(-1)^r(z)^k(1-z)^{jr-k}}{(\alpha+jr+ak+\ell+1)}\binom{\ell'}{r}\binom{\ell+jr}{k}{}_2F_1\left[\begin{matrix}1,\ 1-c-\quad c-\quad c+\quad -jcr,\\ (\alpha+1+jr+ak+b+\ell)/b,\end{matrix}\ y\right]. \tag{2.1}$$

b.
$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{\xi_1^k\xi_2^p\binom{\alpha+ak+k+bp}{k}\binom{\beta+ack+bcp+p}{p}}{(\beta+ack+bcp+\ell+1)\binom{(\beta+1+\ell+ack+bcp+j\ell')/j}{\ell'}}$$
$$= (1-y)^{\beta+1+\ell}(1-z)^{\alpha+1} \times$$
$$\sum_{r=0}^{\ell'}\sum_{p=0}^{\ell+jr}\frac{(-1)^r(y)^p(1-y)^{jr-p}}{(\beta+\ell+1+jr+bcp)}\binom{\ell'}{r}\binom{\ell+jr}{p}{}_2F_1\left[\begin{matrix}1,\ (c-\beta-1+ac-jr-\ell)/c,\\ (\ell+\beta+1+jr+bcp)/(ac),\end{matrix}\ z\right], \tag{2.2}$$

where $\binom{a}{n} = \dfrac{\Gamma(a+1)}{n!\,\Gamma(a-n+1)}$, and y and z are defined through

$$\xi_1 = \frac{-z}{(1-z)^{a+1}(1-y)^{ac}},\quad \xi_2 = \frac{-y}{(1-z)^b(1-y)^{bc+1}}$$
$$|y| < 1,\ |z| < 1,\ |\xi_1| < 1,\ \text{and } |\xi_2| < 1.$$

Corollary 1a: Reduction of Theorem 1a for the Jacobi polynomial gives:

$$\sum_{n=0}^{\infty}\frac{\xi^n}{(\sigma+\tau+1+\ell+an+bn+n)\binom{\frac{\sigma+\tau+1+\ell+an+bn+n+j}{j}}{\ell'}}P_n^{(\sigma+an,\,\tau+bn)}(w)$$
$$= \sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\frac{(-z)^k(1-z)^{\ell+jr-k+\sigma+\tau+1}(1-y)^{-\sigma}(-\ell')_r(-\ell-jr)_k}{k!\,r!\,\ell'!\,(\ell+jr+(1+a+b)k+\sigma+\tau+1)} \times$$
$${}_2F_1\left[\begin{matrix}1,\ \dfrac{(1+a)(1+\ell+\tau+jr)-\sigma b}{1+a+b},\\ \dfrac{\ell+jr+(1+a+b)k+\sigma+\tau+a+b+2}{1+a+b},\end{matrix}\ y\right]. \tag{2.3}$$

$|y| < 1$, $|z| < 1$, and $|\xi| < 1$, where $(a+1)_n/n! = \binom{a+n}{n}$, and y and z are defined in (1.5).

Corollary 1b: Reduction of Theorem 1b for the Jacobi polynomial gives:

$$\sum_{n=0}^{\infty} \frac{\xi^n}{(\sigma + n(1+a) - \ell)\left(\dfrac{\ell - \sigma - (1+a) + j}{j}\right)_{\ell'}} P_n^{(\sigma+an,\,\tau+bn)}(w)$$

$$= \frac{(1-z)^{\sigma+\tau+1}}{\ell'!}(1-y)^{-\sigma+\ell} \sum_{r=0}^{\ell'} \sum_{p=0}^{\ell+jr} \frac{(-y)^p (1-y)^{jr+p}(-\ell')_r(-\ell-jr)_p}{p!\,r!\,(\sigma + p(1+a) - \ell - jr)} \times$$

$${}_2F_1\left[\begin{matrix} 1, \dfrac{(1+a)(1+\tau+\ell+jr) - b(\sigma-\ell-jr)}{(1+a)}, \\ \dfrac{\sigma+(1+a)p-\ell-jr+a+1}{(1+a)}, \end{matrix}\; z\right], \tag{2.4}$$

where y and z are defined in (1.5).

Proof of Theorem 1a: Now consider the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \int_0^1 x^{c\ell}(1-x^{jc})^{\ell'}\, \delta^n [x^{\alpha c+nc-\beta} D^m\{(1-x^{ac})^n(1-x^{bc})^n x^{\beta+m}\}]dF \tag{2.5}$$

$$\left(\text{where } F \equiv x^c,\ D \equiv \frac{d}{dF},\ \delta \equiv \frac{d}{dF}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \sum_{r=0}^{\ell'} \frac{(-\ell')_r(-\ell-jr)_n}{r!} \int_0^1 x^{c\ell+jcr+\alpha c-\beta} D^m[(1-x^{ac})^n(1-x^{bc})^m x^{\beta+m}]dF \tag{2.6}$$

where $(a)_m = \Gamma(a+m)/\Gamma(a)$, quotient of gamma functions. Equation (2.6) is deduced from (2.5) by expanding $(1-x^{cj})^{\ell'}$ and integrating the resulting equation by parts n times. Equation (2.6) may, in turn, by reduced to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n y^m}{n!\,m!} \sum_{r=0}^{\ell'} \frac{(-\ell')_r(-\ell-jr)_n(-n)_k(-m)_p(\beta+1+ack+bcp)_m}{r!\,k!\,p!\,(\ell+1+jr+\alpha+ak+bp)} \tag{2.7}$$

The evaluation is achieved through a further integration by parts m times, expansions of $(1-x^{ac})^n$ and $(1-x^{bc})^m$, and subsequent integration. By applying the double series transform to (2.7), one obtains

$$\sum_{r=0}^{\ell'}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-z)^k(-y)^p z^n y^m(-\ell')_r(-\ell-jr+k)_n(-\ell-ar)_k(\beta+1+p+bcp+ack)_m}{n!\,m!\,k!\,p!\,r!\,(\ell+\alpha+1+jr+ak+bp)} \tag{2.8}$$

$$= \sum_{r=0}^{\ell'}\sum_{k=0}^{\ell+jr}\sum_{p=0}^{\infty} \frac{(-z)^k(-y)^p(-\ell')_r(-\ell-jr)_k\Gamma(\beta+1+p+bcp+ack)(1-z)^{\ell+jr-k}}{r!\,k!\,p!\,\Gamma(\beta+1+ack+bcp)(\ell+\alpha+1+jr+ak+bp)(1-y)^{\beta+1+p+bcp+ack}} \tag{2.9}$$

We now return to our original expression (2.5) and proceed with its evaluation through a modified approach. Consider the operator and its expansion:

$$\delta^n[x^{\alpha c+nc-\beta}D^m\{(1-x^{ac})^n(1-x^{bc})^n x^{\beta+m}\}]$$

$$= \sum_{p=0}^{m}\sum_{k=0}^{n} \frac{(-n)_k(-m)_p\Gamma(\beta+m+1+ack+bcp)\Gamma(\alpha+n+1+ak+bp)}{k!\,p!\,\Gamma(\beta+1+ack+bcp)\Gamma(\alpha+1+ak+bp)}. \tag{2.10}$$

With the aid of (2.10), (2.5) may be reduced to give:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n y^m}{n!m!}\sum_{m=0}^{p}\sum_{k=0}^{n}\frac{(-n)_k(-m)_p\Gamma(\beta+m+1+ack+bcp)\Gamma(\alpha+n+1+ak+bp)}{k!p!\Gamma(\beta+1+ack+bcp)\Gamma(\alpha+1+ak+bp)}$$

$$\times\ \frac{\ell'!\Gamma[(\alpha+ak+bp+\ell+1)/j]}{(j)\Gamma[(\alpha+ak+bp+\ell+1+j\ell'+j)/j]}. \tag{2.11}$$

Using the double series transformation and reducing the subsequent series over n and m gives:

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-z)^k(-y)^p\Gamma(\beta+1+ack+bcp+p)\Gamma(\alpha+1+ak+k+bp)\Gamma[(\alpha+ak+bp+\ell+1)/j]\ell'!}{k!p!(1-z)^{\alpha+1+ak+k+bp}(1-y)^{\beta+1+ack+bcp+p}\Gamma(\alpha+1+ak+bp)\Gamma(\beta+1+ack+bcp)}$$

$$\times\ \frac{1}{(j)\Gamma[(\alpha+ak+bp+\ell+1+j\ell'+j)/j]}. \tag{2.12}$$

Now equating the expressions (2.9) and (2.12) together with some simple transformations yields the required Theorem 1a.

Proof of Theorem 1b: The procedure adopted is similar to that for Theorem 1a. The modified integral is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n y^m}{n!m!}\int_0^1 x^{\ell+\beta+1-\alpha c-c}(1-x^j)^{\ell'}\delta^n[x^{\alpha c+nc-\beta}D^m\{(1-x^{ac})^n(1-x^{bc})^m x^{\beta+m}\}]dF,$$

where the previous definitions are in effect. The details of the proof follow the proof of Theorem 1a to give expression (2.2).

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