ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-442 *Proposed by P. L. Mana, Albuquerque, NM*

The identity

$$2\cos^2\theta = 1 + \cos(2\theta)$$

leads to the identity

$$8\cos^4\theta = 3 + 4\cos(2\theta) + \cos(4\theta).$$

Are there corresponding identities on Lucas numbers?

B-443 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For all integers n and w with w odd, establish the following

$$L_{n+2w}L_{n+w} - 2L_w L_{n+w} L_{n-w} - L_{n-w}L_{n-2w} = L_n^2(L_{3w} - 2L_w).$$

B-444 *Proposed by Herta T. Freitag, Roanoke, VA*

In base 10, the palindromes (that is, numbers reading the same forward or backward) 12321 and 112232211 are converted into new palindromes using

$$99[10^3 + 9(12321)] = 11077011,$$
$$99[10^5 + 9(112232211)] = 100008800001.$$

Generalize on these to obtain a method or methods for converting certain palindromes in a general base b to other palindromes in base b.

B-445 *Proposed by Wray G. Brady, Slippery Rock State College, PA*

Show that

$$5F_{2n+2}^2 + 2L_{2n}^2 + 5F_{2n-2}^2 = L_{2n+2}^2 + 10F_{2n}^2 + L_{2n-2}^2$$

and find a simpler form for these equal expressions.

B-446 *Proposed by Jerry M. Metzger, University of N. Dakota, Grand Forks, ND*

It is familiar that a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3. The same is true for 9. For 27, this is false since, for example, 27 divides $1 + 8 + 9 + 9$ but does not divide 1899. However, $27 | 1998$.

Prove that 27 divides the sum of the digits of n if and only if 27 divides one of the integers formed by permuting the digits of n.

B-447 *Based on the previous proposal by Jerry M. Metzger.*

Is there an analogue of B-446 in base 5?

SOLUTIONS

Consequence of the Euler-Fermat Theorem

B-418 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that $n^{15} - n^3$ is an integral multiple of $2^{15} - 2^3$ for all integers n.

Solution by Lawrence Somer, Washington, D.C.

The assertion is correct. First, note that

$$n^{15} - n^3 = n^3(n^{12} - 1).$$

Further,

$$2^{15} - 2^3 = 2^3(2^6 - 1)(2^6 + 1) = 8(9)(7)(5)(13).$$

By Euler's generalization of Fermat's theorem,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

if $(a, n) = 1$, where ϕ is Euler's totient function. It follows that $a^{k\phi(d)} \equiv 1 \pmod{d}$ for integral k. Now

$$\phi(8) = 4,\ \phi(9) = 6,\ \phi(7) = 6,\ \phi(5) = 4, \text{ and } \phi(13) = 12.$$

Thus, it follows in each instance that if $(n, d) = 1$, where $d = 8, 9, 7, 5$, or 13, then $n^{12} - 1 \equiv 0 \pmod{d}$, since $\phi(d)|12$ for each d. Further, if $(n, d) \neq 1$ for $d = 8, 9, 7, 5$, or 13, then $d|n^3$, since $d|p^3$ for some prime p. Since $(8, 9, 7, 5, 13) = 1$, it now follows that

$$n^3(n^{12} - 1) \equiv 0 \pmod{8 \cdot 9 \cdot 7 \cdot 5 \cdot 13}.$$

Thus, $2^{15} - 2^3$ divides $n^{15} - n^3$.

Also solved by Paul S. Bruckman, Duane A. Cooper, M. J. DeLeon, Robert M. Giuli, Bob Prielipp, C. B. Shields, Sahib Singh, Gregory Wulczyn, and the proposer.

NOTE: DeLeon generalized to show that for $k \in \{2, 3, 4\}$, $2^k(2^{12} - 1)$ divides $n^k(n^{12} - 1)$ for all positive integers n.

Symmetric Congruence

B-419 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For i in $\{1, 2, 3, 4\}$, establish a congruence

$$F_n L_{5k+i} \equiv a_i n L_n F_{5k+i} \pmod{5}$$

with each a_i in $\{1, 2, 3, 4\}$.

Solution by Sahib Singh, Clarion State College, Clarion, PA

We know that $nL_n \equiv F_n \pmod{5}$. (See the solution to Problem B-368 in the December 1978 issue.) Thus

$$F_n = nL_n \pmod{5}, \tag{1}$$

and $$(5k + i)L_{5k+i} \equiv F_{5k+i} \pmod{5} \text{ or } L_{5k+i} \equiv (i)^{-1}F_{5k+i} \pmod{5}. \tag{2}$$

Multiply (1) and (2) to get

$$F_n L_{5k+i} \equiv (i)^{-1} n L_n F_{5k+i} \pmod 5.$$

Thus, $a_i = (i)^{-1}$ where $(i)^{-1}$ is the multiplicative inverse of i in Z_5. Therefore, $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, and $a_4 = 4$.

Also solved by Paul S. Bruckman, M. J. DeLeon, Bob Prielipp, and the proposer.

Finding Fibonacci Factors

B-420 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let

$$g(n, k) = F^4_{n+10k} + F_n^4 - (L_{4k} + 1)(F^4_{n+8k} + F^4_{n+2k}) + L_{4k}(F^4_{n+6k} + F^4_{n+4k}).$$

Can one express $g(n, k)$ in the form $L_r F_s F_t F_u F_v$ with each of r, s, t, u, and v linear in n and k?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The answer to the question stated above is "yes."

On pp. 376-377 of the December 1979 issue (see solution to Problem H-279) Paul Bruckman established that

$$F^4_{n+6k} - (L_{4k} + 1)(F^4_{n+4k} - F^4_{n+2k}) - F_n^4 = F_{2k}F_{4k}F_{6k}F_{4n+12k}.$$

Substituting $n + 4k$ for n yields

$$F^4_{n+10k} - (L_{4k} + 1)(F^4_{n+8k} - F^4_{n+6k}) - F^4_{n+4k} = F_{2k}F_{4k}F_{6k}F_{4n+28k}.$$

Thus, $g(n, k) =$

$$\begin{aligned}
&[F^4_{n+10k} - (L_{4k} + 1)(F^4_{n+8k} - F^4_{n+6k}) - F^4_{n+4k}] \\
&\quad - [F^4_{n+6k} - (L_{4k} + 1)(F^4_{n+4k} - F^4_{n+2k}) - F_n^4] \\
&= F_{2k}F_{4k}F_{6k}F_{4n+28k} - F_{2k}F_{4k}F_{6k}F_{4n+12k} \\
&= F_{2k}F_{4k}F_{6k}[F_{(4n+20k)+8k} - F_{(4n+20k)-8k}] \\
&= F_{2k}F_{4k}F_{6k}F_{8k}L_{4n+20k},
\end{aligned}$$

because $F_{s+t} - F_{s-t} = F_t L_s$, t even (see p. 115 of the April 1975 issue of this journal).

Also solved by Paul S. Bruckman and the proposer.

Unique Representation

B-421 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $\{u_n\}$ be defined by the recursion $u_{n+3} = u_{n+2} + u_n$ and the initial conditions $u_1 = 1$, $u_2 = 2$, and $u_3 = 3$. Prove that every positive integer N has a unique representation

$$N = \sum_{i=1}^{n} c_i u_i,$$

with $c_n = 1$, each $c_i \in \{0, 1\}$, $c_i c_{i+1} = 0 = c_i c_{i+2}$ if $1 \le i \le n - 2$.

Solution by Paul S. Bruckman, Concord, CA

We first observe that the condition "$c_i c_{i+1} = 0 = c_i c_{i+2}$ for $1 \le i \le n - 2$" should be replaced by

$$c_i c_{i+1} = 0 \text{ for } 1 \le i \le n - 1 \quad \text{and} \quad c_i c_{i+2} = 0 \text{ for } 1 \le i \le n - 2. \tag{1}$$

Let $U = (u_n)_{n=1}^{\infty}$. We call a representation $(N)_U \equiv c_n c_{n-1} \ldots c_1$ of N a *U-nary* representation of N if

$$N = \sum_{i=1}^{n} c_i u_i,$$

with the c_i's satisfying the given conditions, as modified by (1). It is not assumed a priori that such a representation is necessarily unique. In any U-nary representation of N, any two consecutive "1's" appearing must be separated by at least two zeros. Without the modification given in (1), the representations are certainly not unique; examples:

$$(3)_U = 100 = 11 \quad \text{and} \quad (11)_U = 11001 = 100010,$$

ignoring (1) and substituting the given condition of the published problem.

We require a pair of preliminary lemmas.

Lemma 1:

$$\sum_{k=0}^{m} u_{n-3k-1} = u \quad - 1, \; (n = 2, 3, 4, \ldots), \text{ where } m = \left[\frac{n-2}{3}\right]. \tag{2}$$

Proof: Using the recursion satisfied by the u_n's,

$$\sum_{k=0}^{m} u_{n-3k-1} = \sum_{k=0}^{m} (u_{n-3k} - u_{n-3k-3}) = \sum_{k=0}^{m} u_{n-3k} - \sum_{k=1}^{m+1} u_{n-3k} = u_n - u_{n-3m-3}.$$

Note that $n - 3m - 3 = -1$, 0, or 1 for all n. We may extend the sequence U to nonpositive indices k of u_k by using the initial values and the recursion satisfied by the elements of U; we then obtain:

$$u_{-1} = u_0 = u_1 = 1.$$

This establishes the lemma.

Lemma 2: If $(u_n)_U = c_m c_{m-1} \ldots c_1$, then $m = n$ and $c_i = \delta_{ni}$ (Kronecker delta).

Proof: By definition,

$$c_m = 1 \quad \text{and} \quad u_n = \sum_{i=1}^{m} c_i u_i.$$

Since $u_n \geq u_m$, thus $m \leq n$. On the other hand, since any two consecutive "1's" in a U-nary representation are separated by at least two zeros, it follows that

$$u_n \leq \sum_{i=0}^{h} u_{m-3k}, \text{ where } h = \left[\frac{m-1}{3}\right].$$

Substituting $n = m + 1$ in Lemma 1, it follows that $u_n \leq u_{m+1} - 1$, or $u_n < u_{m+1}$. Since $u_m \leq u_n < u_{m+1}$, it follows that $m = n$. Hence $c_n = 1$, from which it follows that the remaining c_i's vanish. Q.E.D.

Now, define S to be the set of all positive integers N that have a *unique* U-nary representation. We will find it convenient to extend S to include the number zero. Note that zero certainly satisfies all the conditions of "U-naryness," *except* for $c_n = 1$; for this exceptional element of S only, we waive this condition. Note that $u_k = k \in S$, $k = 1, 2, 3, 4$.

We seek to establish that S consists of all nonnegative integers, and our proof is by induction on k. Assume that $K \in S$, $0 \leq K < u_k$, where $k \geq 4$. In particular, $M \in S$, where $0 \leq M < u_{k-2}$. Then $(M)_U = c_r c_{r-1} \ldots c_1$, for some r, where $c_r = 1$. Since $M < u_{k-2}$, thus $r \leq k - 3$; otherwise, $r \geq k - 2$, which implies $M \geq u_{k-2}$, a contradiction. Let

$$N = M + u_k. \tag{3}$$

Then

$$N = \sum_{i=1}^{k} c_i u_i, \text{ with } c_k = 1,\ c_i = 0, \text{ if } r < i < k.$$

Since $r \le k - 3$, we see that the foregoing expression yields a U-nary representation of N, namely $(N)_U = c_k c_{k-1} \ldots c_1$, though not necessarily unique. Suppose that $(N)_U = d_t d_{t-1} \ldots d_1$ is another U-nary representation of $N = M + u_k$. Then (since $M \in S$) $d_i = c_i$, $1 \le i \le r$. Moreover, $u_k = N - M$ has a unique U-nary representation, by Lemma 2; hence, $t = k$, which implies that $N \in S$.

Since $0 \le M < u_{k-2}$, thus $u_k \le N < u_{k-2} + u_k = u_{k+1}$. The inductive step is:

$$S \supset \{0, 1, 2, \ldots, u^{k} - 1\} \Rightarrow S \supset \{0, 1, 2, \ldots, u_{k+1} - 1\}.$$

By induction, S consists of all nonnegative integers. Q.E.D.

Also solved by Sahib Singh and the proposer.

Lexicographic Ordering of Coefficients

B-422 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

With representations as in B-421, let

$$N = \sum_{i=1}^{n} c_i u_i,\quad N + 1 = \sum_{i=1}^{m} d_i u_i.$$

Show that $m \ge n$ and that if $m = n$ then $d_k > c_k$ for the largest k with $c_k \ne d_k$.

Solution by Paul S. Bruckman, Concord, CA

We refer to the notation and solution of B-421 above. Given

$$(N)_U = c_n c_{n-1} \ldots c_1 \quad\text{and}\quad (N + 1)_U = d_m d_{m-1} \ldots d_1,$$

which we now know are the unique U-nary representations of N and $N + 1$, respectively.

Since $u_n \le N < u_{n+1}$ and $u_m \le N + 1 < u_{m+1}$, thus $u_m - u_{n+1} < 1 < u_{m+1} - u_n$. Now $u_{m+1} > u_n + 1 > u_n \Rightarrow m + 1 > n$, since U is an increasing sequence. On the other hand, $u_m < u_{n+1} + 1 \le u_{n+2} \Rightarrow m < n + 2$. Hence,

$$m = n \quad\text{or}\quad m = n + 1. \tag{1}$$

Note that (1) is somewhat stronger than the desired result: $m \ge n$.

Now, suppose $m = n$, and let k be the largest integer i such that $c_i \ne d_i$. Then $c_i = d_i$, $k < i \le n$. Hence,

$$\sum_{i=k+1}^{n} c_i u_i = \sum_{i=k+1}^{n} d_i u_i.$$

This, in turn, implies

$$N - \sum_{i=k+1}^{n} c_i u_i = N + 1 - 1 - \sum_{i=k+1}^{n} d_i u_i,$$

or

$$1 + \sum_{i=1}^{k} c_i u_i = \sum_{i=1}^{k} d_i u_i.$$

Suppose $c_k = 1$, $d_k = 0$. Then the left member of (2) is $\ge 1 + u_k$. On the other hand, the right member of (2) is

$$\le \sum_{i=0}^{p} u_{k-1-3i} = u_k - 1, \text{ where } p = \left[\frac{k-2}{3}\right],$$

using the properties of the U-nary representation and Lemma 1 of the solution to B-421. This contradiction establishes the only remaining possibility, i.e., $c_k = 0$, $d_k = 1$. This establishes the desired result.

Also solved by Sahib Singh and the proposer.

Telescoping Infinite Product

B-423 *Proposed by Jeffery Shallit, Palo Alto, CA*

Here let F_n be denoted by $F(n)$. Evaluate the infinite product

$$\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{13}\right)\left(1 + \frac{1}{610}\right) \cdots = \prod_{n=1}^{\infty}\left[1 + \frac{1}{F(2^{n+1} - 1)}\right].$$

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Let L_n also be written as $L(n)$ and $A_n = 1 + [1/F(2^{n+1} - 1)]$. It is easily seen (for example, from the Binet formulas) that

$$L(2)L(4)L(8) \ldots L(2^n) = F(2^{n+1}) \quad \text{and} \quad 1 + F(2^{n+1} - 1) = F(2^n - 1)L(2^n).$$

Hence, $A_n = F(2^n - 1)L(2^n)/F(2^{n+1} - 1)$ and

$$\prod_{i=1}^{\infty} A_n = \lim_{n\to\infty} \frac{F(1)F(3)F(7)F(15) \cdots F(2^n - 1)L(2)L(4)L(8) \cdots L(2^n)}{F(3)F(7)F(15) \cdots F(2^{n+1} - 1)}$$

$$= \lim_{n\to\infty} \frac{F(2^{n+1})}{F(2^{n+1} - 1)},$$

and the desired limit is $\alpha = (1 + \sqrt{5})/2$.

Also solved by Paul S. Bruckman, Bob Prielipp, and the proposer.

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(Continued from page 6)

Hence

$$u_{n-1} = x_1 u_n - Dy_1 v_n = (x_1 - 1)u_n - Dy_1 v_n + u_n \geq u_n.$$

Thus $n = 0$.

REFERENCES

1. M. J. DeLeon. "Pell's Equation and Pell Number Triples." *The Fibonacci Quarterly* 14 (Dec. 1976):456-460.
2. Trygve Nagell. *Introduction to Number Theory.* New York: Chelsea, 1964.

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