Finally, we note the following interesting fact. Since

$$a_0(r) = \pm\frac{1}{r+1}$$

and

$$S_0(n) = n,$$

it follows from (2) that

$$S_r(n) = S_1(n)P_{r-1}(n),$$

where $P_{r-1}(n)$ is a polynomial in n of degree $r - 1$.

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A NOTE ON THE POLYGONAL NUMBERS

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1. INTRODUCTION

Polygonal numbers of order k ($k = 3, 4, 5, \ldots$) are the numbers

$$P_{n,k} = \frac{1}{2}[(k-2)n^2 - (k-4)n] \quad (n = 1, 2, 3, \ldots). \tag{1}$$

If $k = 4$, they are reduced to the square numbers. It is clear that there are an infinite number of square numbers which are at a time the sum and difference and the product of such numbers, from the identity

$$\begin{aligned}(4m^2+1)^2 &= (4m)^2 + (4m^2-1)^2 \\ &= (8m^4+4m^2+1)^2 - (8m^4+4m^2)^2,\end{aligned}$$

and since there are an infinite number of composite numbers of the form $4m^2+1$ (for example, if $m = 5j + 1$, $4m^2+1$ is divisible by 5).

Sierpinski [1] proved that there are an infinite number of triangular numbers ($k = 3$) which are at a time the sum and the difference and the product of such numbers.

For $k = 5$, Hansen [2] proved that there are an infinite number of $P_{n,5}$ that can be expressed as the sum and the difference of such numbers.

O'Donnell [3] proved a similar result for $k = 6$, and conjectured that there will be a similar result for the general case.

In this paper it will be shown that their method of proof is valid for the general case, proving the following theorem.

Theorem: Let a and b be given integers such that $a \neq 0$ and $a \equiv b \pmod 2$, and let

$$A_n = \frac{1}{2}(an^2 + bn) \quad (n = 1, 2, 3, \ldots). \tag{2}$$

There are an infinite number of A_n's which can be expressed as the sum and the difference of the numbers of the same type.

2. PROOF OF THE THEOREM

If $a < 0$, we obtain a set of integers whose elements are the negatives of the elements in the set obtained by using $-a$ and $-b$ instead of a and b. Hence we can assume $a > 0$ in the following.

Let

(3) $$B_n = A_n - A_{n-r} = \frac{1}{2}[a(2nr - r^2) + br],$$

where n and r are positive integers, $n > r$, and r is odd unless a is even.

Lemma 1: For

(4) $$m = ars + r,$$

where s is a positive integer such that

(5) $$a^2s + 2a > -\frac{b}{r},$$

the equation

(6) $$A_m = B_n = A_n - A_{n-r}$$

is satisfied by the integer

(7) $$n = \frac{1}{2}s[r(a^2s + 2a) + b] + r.$$

Proof: Solving

$$\frac{1}{2}[ar^2(as + 1)^2 + br(as + 1)] = \frac{1}{2}[a(2nr - r^2) + br]$$

for n, we have (7).

For any integer c, $c^2 \equiv c \pmod 2$, so that

$$s[r(a^2s + 2a) + b] = ra^2s^2 + 2ars + bs \equiv ras + as$$
$$= (r + 1)as \equiv 0 \pmod 2,$$

by the conditions for r and a, which ensures that n is an integer, and the lemma is proved.

For m and n of Lemma 1,

(8) $$A_n = A_m + A_{n-r}.$$

In order to find a number of this type which is equal to some B_p, let $s = art$, for any positive integer t such that

(9) $$a^3r^2t + b \geq 0.$$

Then (5) is satisfied and from (4) and (7) we have

(10) $$m = a^2r^2t + r,$$

(11) $$n = aru + r,$$

where

(12) $$u = \frac{1}{2}t[r(a^3rt + 2a) + b]$$

is an integer such that $u \geq s$ by the condition (9).

From Lemma 1, using u in place of s, for the integer

$$p = \frac{1}{2}u[r(a^2u + 2a) + b] + r, \tag{13}$$

we have $A_n = B_p$. This equation, together with equation (8), provides the following lemma, from which we can easily establish the theorem.

Lemma 2: Let a, r, and t be positive integers, where r is odd unless a is even and the condition (9) is satisfied. Then, m, n, u, and p, which are given by (10), (11), (12), and (13), respectively, are also positive integers, and

$$A_n = A_m + A_{n-r} = A_p - A_{p-r}.$$

3. THE CASE OF POLYGONAL NUMBERS

The result for the polygonal numbers of order k is given for

$$a = k - 2, \quad b = -(k - 4)$$

in Lemma 2. In this case, condition (9) is always satisfied for any positive integer t.

Example 1: For $r = 1$, we have

$$P_{n,k} = P_{m,k} + P_{n-1,k} = P_{p,k} - P_{p-1,k},$$

where

$$m = (k - 2)^2t + 1,$$

$$n = (k - 2)u + 1,$$

and

$$p = \frac{1}{2}u[(k - 2)^2u + k] + 1$$

for

$$u = \frac{1}{2}t[(k - 2)^3t + k].$$

Let T_n, Q_n, P_n, H_n, and S_n denote $P_{n,k}$ for $k = 3, 4, 5, 6,$ and 7, respectively. Then we have

$$T_{\frac{1}{2}(t^2+3t)+1} = T_{t+1} + T_{\frac{1}{2}(t^2+3t)} = T_p - T_{p-1},$$

where $p = \frac{1}{8}(t^4 + 6t^3 + 15t^2 + 18t) + 1$,

$$Q_{8t^2+4t+1} = Q_{4t+1} + Q_{8t^2+4t} = Q_p - Q_{p-1},$$

where $p = 32t^4 + 32t^3 + 16t^2 + 4t + 1$,

$$P_{\frac{1}{2}(81t^2+15t)+1} = P_{9t+1} + P_{\frac{1}{2}(81t^2+15t)} = P_p - P_{p-1},$$

where $p = \frac{1}{8}(6561t^4 + 2430t^3 + 495t^2 + 50t) + 1$,

$$H_{128t^2+12t+1} = H_{16t+1} + H_{128t^2+12t} = H_p - H_{p-1},$$

where $p = 8192t^4 + 1536t^3 + 168t^2 + 9t + 1$,

$$S_{\frac{1}{2}(625t^2+35t)+1} = S_{25t+1} + S_{\frac{1}{2}(625t^2+35t)} = S_p - S_{p-1},$$

where $p = \frac{1}{8}(390625t^4 + 43750t^3 + 2975t^2 + 98t) + 1$.

Example 2: For the case $r = 3$, $t = 1$, we have

$$m = 9(k - 2)^2 + 3,$$
$$n = 3(k - 2)u + 3,$$

and

$$p = \frac{1}{2}u[3(k - 2)^2u + 5k - 8] + 3,$$

where

$$u = \frac{1}{2}(9k^3 - 54k^2 + 113k - 80).$$

For $k = 6$, it gives

$$H_{3591} = H_{147} + H_{3588} = H_{2148916} - H_{2148913},$$

which is not covered by Theorem 2 of O'Donnell [3].

The generalized relation in Lemma 2, however, does not yield all such relations. For instance, the relation

$$H_{25} = H_{10} + H_{23} = H_{307} - H_{306}$$

cannot be deduced from our Lemma 2.

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