FIBONACCI NUMBERS AND STOPPING TIMES

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For each integer $k \geq 2$, let $\{a_{n,k}\}$ and $\{b_{n,k}\}$ be two sequences of integers defined by $a_{n,k} = 0$ for all $n = 1, \ldots, k-1$, $a_{k,k} = 1$, and

$$a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}$$

for all $n \geq k$; $b_{1,k} = 0$, and

$$b_{n,k} = a_{n,k} + \sum_{j=1}^{n-1} a_{j,k} b_{n-j,k}$$

for all $n \geq 2$.

Let $\{Y_n\}$ be the fair coin-tossing sequence, i.e.,

$$P(Y_j = 0) = \frac{1}{2} = P(Y_j = 1)$$

for all $j = 1, 2, \ldots$, and $Y_1, Y_2, \ldots$ are independent. With respect to the sequence $\{Y_n\}$, for each integer $k \geq 1$, let $\{R_{n,k}\}$ and $\{N_{n,k}\}$ be two sequences of stopping times defined by

$$R_{1,k}(Y_1, Y_2, \ldots) = \inf\{m \mid Y_m = \cdots = Y_{m-k+1} = 0\},$$

$= \infty$ if no such m exists, and for all $n \geq 2$,

$$R_{n,k}(Y_1, Y_2, \ldots) = \inf\{m \mid m \geq R_{n-1,k} + k \text{ and } Y_m = \cdots = Y_{m-k+1} = 0\},$$

$= \infty$ if no such m exists; $N_{1,k} = R_{1,k}$ and $N_{n,k} = R_{n,k} - R_{n-1,k}$ for all $n \geq 2$.

In this note, we shall prove the following interesting theorems.

Theorem 1: For each integer $k \geq 2$,

$$a_{n,k} = 2^n P(N_{1,k} = n) \quad \text{and} \quad b_{n,k} = 2^n P(R_{m,k} = n \text{ for some integer } m \geq 1).$$

Theorem 2: For each integer $k \geq 2$,

$$b_{n,k} = 2b_{n-1,k} + 1 \quad \text{or} \quad 2b_{n-1,k} - 1 \quad \text{or} \quad 2b_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k-1$.

Theorem 3: For each integer $k \geq 2$, let

$$\mu_k = \sum_{n=1}^{\infty} n 2^{-n} a_{n,k} = E(N_{1,k}),$$

then

$$b_{nk,k} = \{2^{nk} + 2^k - 2\}/\mu_k \quad \text{and} \quad b_{nk+j,k} = 2^{j-1}\{2^{nk+1} - 2\}/\mu_k$$

for all $n \geq 1$ and $j = 1, 2, \ldots, k-1$.

We start with the following elementary lemmas.

Lemma 1: For each integer $k \geq 1$, let

$$\Phi_k(t) = E(tN_{1,k}) \text{ if } E(|t|N_{1,k}) < \infty;$$

then

$$\Phi_k(t) = \left(\frac{t}{2}\right)^k \Big/ \left\{1 - \sum_{j=1}^{k} \left(\frac{t}{2}\right)^j\right\} \quad \text{for all } -1 \leq t \leq 1.$$

Proof: For $k = 1$, it is well known that

$$\Phi_1(t) = \left(\frac{t}{2}\right) \bigg/ \left\{1 - \left(\frac{t}{2}\right)\right\}$$

for all t in $[-1, 1]$. For $k \geq 2$, it is easy to see that $N_{1,k} = N_{1,k-1} + Z$, where Z is a random variable such that

$$P(Z = 1) = P(Z = 1 + N_{1,k}) = \frac{1}{2}$$

and Z is independent of $N_{1,k-1}$. Hence

$$\Phi_k(t) = \frac{t}{2}\{\Phi_{k-1}(t) + \Phi_k(t)\}$$

for all $-2 < t < 2$. Therefore, for each integer $k \geq 1$,

$$\Phi_k(t) = \left(\frac{t}{2}\right) \bigg/ \left\{1 - \sum_{j=1}^{k} \left(\frac{t}{2}\right)\right\}$$

for all $-1 \leq t \leq 1$.

Lemma 2: For each integer $k \geq 2$, let

$$G_k(t) = \sum_{n=1}^{\infty} t^n a_{n,k}$$

for all t such that

$$\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty;$$

then

$$G_k(t) = \Phi_k(2t) \text{ for all } -\frac{1}{2} < t < \frac{1}{2} \text{ and } k \geq 2.$$

Proof: Since $a_{n,k} = 0$ for all $n = 1, 2, \ldots, k - 1$, $a_{k,k} = 1$, and

$$a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}$$

for all $n > k$,

$$G_k(t) = \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i G_k(t).$$

Therefore,

$$G_k(t) = t^k \bigg/ \left\{1 - \sum_{j=1}^{k} t^j\right\}$$

for all $k \geq 2$ and all t such that

$$\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty.$$

Since $a_{n,k} \leq 2^n$ for all $n \geq 1$ and all $k \geq 2$, $G_k(t)$ exists for all $-\frac{1}{2} < t < \frac{1}{2}$. By Lemma 1, we have

$$G_k(t) = \Phi_k(2t) \text{ for all } t \text{ in the interval } \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and all } k \geq 2.$$

For each integer $k \geq 1$, let $u_{0,k} = 1$, and for all $n \geq 1$, let

$$u_{n,k} = P\{R_{m,k} = n \text{ for some integer } m \geq 1\}$$

and $f_{n,k} = P\{N_{1,k} = n\}$. Since $\{Y_n\}$ is a sequence of i.i.d. random variables, and $u_{0,k} = 1$, it is easy to see that

$$u_{n,k} = \sum_{j=1}^{n} f_{j,k} u_{n-j,k} \text{ for all } n \geq 1 \text{ and all } k \geq 1.$$

Hence we have the following theorem.

Theorem 1': For each integer $k \geq 2$, $2^n u_{n,k} = 2^n P\{R_{m,k} = n$ for some integer $m \geq 1\} = b_{n,k}$ and $2^n f_{n,k} = 2^n P\{N_{1,k} = n\} = a_{n,k}$ for all $n \geq 1$.

Let $A = \{(w_1, w_2, \ldots, w_n) | w_i = 0$ or 1 for all $i = 1, 2, \ldots, n$ and $w_j = 1 \neq w_{j+1} = \cdots = w_n = 0$ for some $j = n - Jk$ and some integer $J \geq 1\}$.

Let $B = \{(v_1, v_2, \ldots, v_{n-1}) | v_i = 0$ or 1 for all $i = 1, 2, \ldots, n-1$ and $v_{j-1} = 1 \neq v_j = \cdots = v_{n-1} = 0$ for some $j = n - Jk$ for some integer $J \geq 1\}$.

Lemma 3: For each integer $k \geq 2$,

$$2^n u_{n,k} = 2^n u_{n-1,k} + 1 \quad \text{or} \quad 2^n u_{n-1,k} - 1 \quad \text{or} \quad 2^n u_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k - 1$.

Proof: By the definition of $\{u_{n,k}\}$, for each integer $k \geq 2$,

$$2^n u_{n,k} = \text{the number of elements in } A$$

and

$$2^{n-1} u_{n-1,k} = \text{the number of elements in } B.$$

(i) If $n = mk$ for some integer $m \geq 1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if $(v_1, v_2, \ldots, v_{n-1})$ is in B, and $(0, 0, \ldots, 0)$, n-tuple, is also in A even $(0, 0, \ldots, 0)$, $(n - 1)$-tuple, is not in B. Hence the number of elements in $A \geq 2 \cdot$ the number of elements in $B + 1$. Since each element $(w_1, w_2, \ldots, w_n)$ in A such that $w_j \neq w_{j+1}$ for some $1 \leq j \leq n-1$ is a form of $(0, v_1, v_2, \ldots, v_{n-1})$ or a form of $(1, v_1, v_2, \ldots, v_{n-1})$ for some element $(v_1, v_2, \ldots, v_{n-1})$ in B. Hence the number of elements in $A \leq 2 \cdot$ the number of elements in $B + 1$. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in $B + 1$.

(ii) If $n = mk + 1$ for some integer $m \geq 1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if $(v_1, v_2, \ldots, v_{n-1})$ is in B and $v_j \neq v_{j+1}$ for some $1 \leq j \leq n-2$ and $(1, 0, 0, \ldots, 0)$, n-tuple, is also in A [$(0, 0, \ldots, 0)$, $(n - 1)$-tuple, is in B]. Hence the number of elements in $A \geq 2 \cdot$ the number of elements in $B - 1$. Since each element $(w_1, w_2, \ldots, w_n)$ in A such that $w_j \neq w_{j+1}$ for some $2 \leq j \leq n - 1$ is a form of $(0, v_1, v_2, \ldots, v_{n-1})$ or a form of $(1, v_1, v_2, \ldots, v_{n-1})$ for some element $(v_1, v_2, \ldots, v_{n-1})$ in B. Hence the number of elements in $A \leq 2 \cdot$ the number of elements in $B - 1$. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in $B - 1$.

(iii) If $n = mk + j$ for some integers $m \geq 1$ and $2 \leq j \leq k-1$, then $(0, v_1, v_2, \ldots, v_{n-1})$ and $(1, v_1, v_2, \ldots, v_{n-1})$ are in A if and only if $(v_1, v_2, \ldots, v_{n-1})$ is in B. Therefore, the number of elements in $A = 2 \cdot$ the number of elements in B.

By (i), (ii), and (iii), the proof of Lemma 3 is now complete.

Theorem 2': For each integer $k \geq 2$,

$$b_{n,k} = 2b_{n-1,k} + 1 \quad \text{or} \quad 2b_{n-1,k} - 1 \quad \text{or} \quad 2b_{n-1,k}$$

according as $n = mk$ or $mk + 1$ or $mk + j$ for some integers $m \geq 1$ and $j = 2, 3, \ldots, k - 1$.

Proof: By Theorem 1' and Lemma 3.

For each integer $k \geq 1$, let

$$\mu_k = E\{N_{1,k}\} = \sum_{n=1}^{\infty} nP(N_{1,k} = n) = \sum_{n=1}^{\infty} nf_{n,k}.$$

By Theorem 1',

$$\mu_k = \sum_{n=1}^{\infty} n2^{-n}a_{n,k} \text{ for each integer } k \geq 2.$$

Since

$$\left(\frac{1}{2}\right)^k = \sum_{j=0}^{k-1} u_{n-j,k}\left(\frac{1}{2}\right)^j \text{ for all } n \geq k \text{ and } k \geq 1,$$

$$\left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \sum_{j=0}^{k-1} u_{n-j,k}\left(\frac{1}{2}\right)^j.$$

By the Renewal Theorem (see [1, p. 330]), we have

$$\left(\frac{1}{2}\right)^k = \{E(N_{1,k})\}^{-1}\sum_{j=0}^{k-1}\left(\frac{1}{2}\right)^j \text{ for all } k \geq 1.$$

Hence

$$\mu_k = E\{N_{1,k}\} = \sum_{n=1}^{\infty} nf_{n,k} = \sum_{n=1}^{\infty} n2^{-n}a_{n,k} = \sum_{j=1}^{k} 2^j = 2^{k+1} - 2.$$

Theorem 3': For each integer $k \geq 2$, let

$$\mu_k = \sum_{n=1}^{\infty} n2^{-n}a_{n,k} = 2^{k+1} - 2;$$

then

$$b_{mk,k} = \{2^{mk} + 2^k - 2\}/\mu_k \quad \text{and} \quad b_{mk+j,k} = 2^{j-1}\{2^{mk+1} - 2\}/\mu_k$$

for all integers $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

Proof: By the definition of $\{b_{n,k}\}$, $b_{k,k} = 1$. Hence, by Theorem 2', Theorem 3' holds when $m = 1$. Suppose that Theorem 3' holds for $m = 1, 2, \ldots, M-1$ and $j = 1, 2, \ldots, k-1$, where M is an integer ≥ 2. Now, let $m = M$, then, by Theorem 2',

$$\begin{aligned} b_{Mk,k} = 2b_{Mk-1,k} + 1 &= 2^{k-1}\{2^{(M-1)k+1} - 2\}/\mu_k + 1 = (2^{Mk} - 2^k + 2^{k+1} - 2)/\mu_k \\ &= (2^{Mk} + 2^k - 2)/\mu_k, \end{aligned}$$

since $\mu_k = 2^{k+1} - 2$.

$$b_{Mk+1,k} = 2b_{Mk,k} - 1 = (2^{Mk+1} + 2^{k+1} - 4)/\mu_k - 1 = (2^{Mk+1} - 2)/\mu_k.$$

$$b_{Mk+j,k} = 2^{j-1}b_{Mk+1,k} = 2^{j-1}(2^{Mk+1} - 2)/\mu_k, \text{ for all } j = 2, 3, \ldots, k-1.$$

Hence Theorem 3' holds for $m = M$ and $j = 1, 2, \ldots, k-1$. Therefore, Theorem 3' holds for all $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

Corollary to Theorem 3': For each integer $k \geq 2$,

$$u_{mk,k} = \mu_k^{-1}\{1 + 2^{-mk+k} - 2^{-mk+1}\} = (2^{k+1} - 2)^{-1}\{1 + 2^{-mk+k} - 2^{-mk+1}\}$$

and

$$u_{mk+j,k} = \mu_k^{-1}\{1 - 2^{-mk} = (2^{k+1} - 2)^{-1}\{1 - 2^{-mk}\}$$

for all integers $m \geq 1$ and $j = 1, 2, \ldots, k-1$.

REFERENCE

1. W. Feller. *An Introduction to Probability Theory and Its Applications*. I, 3rd ed. New York: John Wiley & Sons, Inc., 1967.

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