$$ew^k = b_{k+1}w + (b_1 + b_{k+1})w^{k+1} \tag{15}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$ew^{2k-1} = b_1 + b_{k+1} + (b_1 + 2b_{k+1})w^k.$$

The norm of e is ± 1 since $e = w^{-(k-1+kn)}$ and w is a unit. We observe, however, that D is just the transpose of the matrix from which the norm of e was calculated. Hence, det $D = \pm 1$, and our theorem is proved.

As a concluding note we remark that, if $k = 2$, then the theorem yields—with the appropriate choice of the plus/minus signs—the identity

$$F_n = (-1)^{n+1}F_{n+2}^3 + 2(-1)^n F_{n+1}F_{n+2}^2 + (-1)^{n+1}F_{n+1}^3. \tag{16}$$

This can also be verified as follows: Replace F_{n+2} by $F_n + F_{n+1}$ in (16) and simplify to obtain

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \tag{17}$$

Finally, compare (17) with the known [6, p. 57] identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

to complete the verification of (16).

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POINTS AT MUTUAL INTEGRAL DISTANCES IN S^n

B. GLEIJESES

Istituto di Matematica Applicata, Via Belzoni, 7 I35100 Padova, Italy

In radio-astronomy circles, it is sometimes jokingly speculated whether it is possible to place infinitely many in-phase, nonaligned antennas in a plane (say, vertical dipoles in a horizontal plane). Geometrically, this means placing infinitely many nonaligned points in R^2, with integral pairwise distances; and naturally the mathematician wants to generalize to R^3 and R^n. In R^3 there is still a physical meaning for acoustic radiators, but not for electromagnetic radiators, since none exists with a spherical symmetry radiation pattern (for more serious questions on antenna configurations, see [2]).

A slightly different problem is that of placing a receiving antenna in a point P, where it receives in phase from transmitting antennas placed in nonaligned coplanar points A_1, A_2, ... (in phase with each other); geometrically, this means that the distances A_iA_j are integral, and that all the differences PA_i-PA_j are also integral. We shall prove that the first question has a negative answer, and that only finitely many P's satisfy the second condition.

Our proof of these facts, set out in Paragraph 1, is only the first step in an inductive demonstration (given in Sections 1 to 4) of the following

Theorem: In a Euclidean space of dimension $n \geq 2$ there exist only finite sets of noncollinear points all of whose mutual distances are integers.

If in all the reasoning used in Paragraphs 1-4 to prove the theorem, one requires that m be a positive real number rather than a positive integer, then one obtains the following result.

Theorem (bis): If one places $n + 1$ antennas in phase at the vertices of a nondegenerate $(n+1)$-hedron in a Euclidean space of dimension $n \geq 2$, then the set of points of the space from which the signals are received in phase is a finite set.

Remark: Two antennas are in phase if their distance is a multiple of the wavelength; a point P receives in phase from two antennas A and B if the differences AP-BP is a multiple of the wavelength.

The last section describes two methods (one due to Euler) to construct systems of points in the plane with integral mutual distances.

By PQ we denote, as usual, the distance between the points P and Q. The phrase "points at integral distance" will be abbreviated to "points at ID."

1. Let O and A be two points of the plane having distance $OA = a$, an integer. We show that the points of the plane for which OP and AP are both integers must all lie on a distinct hyperbolas (one of which is degenerate). As our coordinate system, we take the orthogonal axes with O as origin and the line through O and A as x-axis. Let P be a point at ID from O and from A, assume $P \neq O$, A, and set $OP = m > 0$, $AP = m - k$ with m and k integers.

Note that by the triangle inequality we have $AP \leq OA + OP$ and $OP \leq OA + AP$, which imply that $-a \leq k \leq a$. It is immediate that P lies on the hyperbola $\mathfrak{a}_k$ with foci at O and A defined by the equation

$$(a^2 - k^2)(x - a/2)^2 - k^2y^2 = (k^2/4)(a^2 - k^2); \tag{1}$$

its center is $A' = (a/2, 0)$, and its axes are the lines $y = 0$ and $x = a/2$. It intersects the x-axis at the points with abscissas $x = a/2 \pm k/2$. We conclude that any point that has integral distance from both O and A must lie on one of the hyperbolas $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_a$.

Note that for $k = 0$ and $k = \pm a$, Eq. (1) defines a degenerate parabola, and that there are $a+1$ curves in all. However, all of the curves will be hyperbolas, exactly a in number, if we take $(x - a/2)y = 0$ as $\mathfrak{a}_a$. See Figure 1.

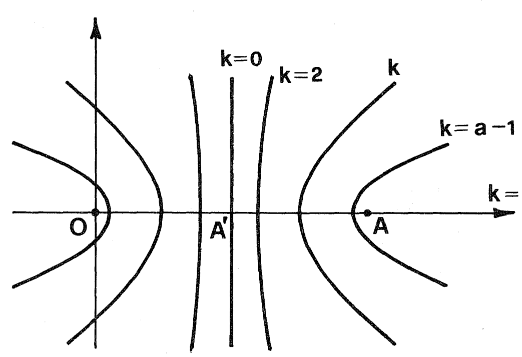

Fig. 1

Now let B be a point at distance $OB = b$ from O and noncollinear with O and A. Repeating the discussion above for A we find that the points at integral distance from O and from B all must lie on b hyperbolas $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_b$. All these hyperbolas have as center the midpoint B' of the segment OB and as axes the line through O and B and the line through B' perpendicular to it. Hence it is clear that none of these hyperbolas coincides with any $\mathfrak{A}_i$, since they have different sets of axes. The possible points at integral distance from O, A, and B are found in the union with respect to i and j of the $\mathfrak{A}_i \cap \mathfrak{B}_j$, and hence are in number at most $4ab$.

2. We now give the proof of the theorem stated in the introduction for the case of a space of dimension 3, since the proof still has an intuitive geometric meaning that sheds light on the more general situation.

Let X be a set of points in the space that are noncollinear and mutually at ID. We wish to show that X has finite cardinality.

If the points of X all lie in a plane, we are already done. Otherwise, X contains four noncoplanar points O, A, B, and C. We set $OA = a$, $OB = b$, and $OC = c$. We fix an orthogonal coordinate system having origin at O, the line OA as x-axis, and the plane determined by OAB as xy plane. With an argument similar to that used in the case of the plane, one sees immediately that the points P at ID from O and A, and thus in particular the points of the set X, must all lie on the $a + 1$ quadrics $S_{A,k}$ defined by the equations

$$S_{A,k}\colon \quad (a^2 - k^2)(x - a/2)^2 - k^2y^2 - k^2z^2 = (k^2/4)(a^2 - k^2), \; k = 0, 1, \ldots, a.$$

If $0 < k < a$, the quadric $S_{A,k}$ is an elliptic hyperboloid of revolution around the line OA; the point A' (midpoint of the segment OA) is its center, and each plane of the pencil through OA is a plane of symmetry; among these there is the xy plane. For $k = 0$ the quadric $S_{A,0}$ is the plane $x = a/2$ (counted twice), and for $k = a$ the real points of $S_{A,a}$ are the real points of the line that passes through O and A.

The points at ID from O and B are all to be found on the $b + 1$ quadrics $S_{B,h}$ with $h = 0, 1, \ldots, b$. For $0 < h < b$, the quadric $S_{B,h}$ is an elliptic hyperboloid of revolution around the line OB; its center is at B' (the midpoint of the segment OB), and it has as planes of symmetry all the planes of the pencil through the axis of revolution, among which there is the xy plane. For $h = 0$ the quadric $S_{B,0}$ is a double plane; for $h = b$ the real points of $S_{B,b}$ are the real points on the axis of revolution.

By analogy, the points at ID from O and C are found on $c + 1$ quadrics $S_{C,\ell}$ with $\ell = 0, 1, \ldots, c$. For $0 < \ell < c$, the quadrics $S_{C,\ell}$ are elliptic hyperboloids of revolution around the line OC, and they certainly do not have the xy plane as a plane of symmetry. For $\ell = 0$ the quadric $S_{C,0}$ is the plane through C' (midpoint of the segment OC) which is orthogonal to the line OC, this plane being counted twice. For $\ell = c$ the real points of the quadric $S_{C,c}$ are the points of the line OC.

Since the points of X are at ID from A, B, C, and O, we have

$$X \subseteq \cup(S_{A,k} \cap S_{B,h} \cap S_{C,\ell}) \text{ for } k = 0, \ldots, a;\; h = 0, \ldots, b;\; \ell = 0, \ldots, c.$$

Now if one of the three quadrics that appear in $S_{A,k} \cap S_{B,h} \cap S_{C,\ell}$ is degenerate ($k = 0$, a, or $h = 0$, b, or $\ell = 0$, c), it is clear that the real points of the intersection either are finitely many or lie in a plane. Therefore, the points of X contained in the intersection are finitely many. If none of those quadrics is degenerate, let $\gamma_{k,h}$ be the real intersection of $S_{A,k}$ with $S_{B,h}$, with k and h fixed.

In view of the facts that the $S_{A,k}$ and $S_{B,h}$ are real quadrics, that there are no real lines contained in them, and that we are considering only the real points of the intersection, there are only two possible cases to discuss:

a. $\gamma_{k,h}$ has real points and is irreducible.

b. $\gamma_{k,h}$ splits into two nondegenerate conics with real points.

In case b, a conic being a plane curve, we see that there can be only a finite number of points of the set X which lie on $\gamma_{k,h} \cap S_{C,\ell}$ for every ℓ. In case *a*, if we show that $\gamma_{k,h}$ cannot lie entirely on any $S_{C,\ell}$, then $\gamma_{k,h} \cap S_{C,\ell}$ is a finite set of points for each ℓ, and that will complete our proof.

So suppose that $\gamma_{k,h}$ is irreducible. Since $S_{C,\ell}$ is not symmetric with respect to the xy plane, we can write its equation in the form

$$(\alpha x + \beta y + \delta)z + F(x, y, z^2) = 0, \text{ with } \alpha, \beta, \delta \text{ not all } 0.$$

A simple calculation shows that the pairs of points symmetric with respect to the xy plane and which lie on the quadric are either in the xy plane itself or in the plane $\alpha x + \beta y + \delta = 0$. Since $\gamma_{k,h}$ is symmetric with respect to the xy plane, were it contained entirely in $S_{C,\ell}$ it would have to be contained in one of the two planes just mentioned. But, a quadric does not contain irreducible plane quartic curves.

3. In R^n the proof is similar and is based on induction on the dimension n of the space. Here we give only a sketch of the demonstration.

Let X be a set of points in R^n that are all mutually at ID.

a. It is evident that the points that have integral distance from a point O and from another point P are located on a finite number (equal to $OP + 1$) of quadrics $S_{P,k}$ with $k = 0, 1, \ldots, OP$. The quadrics $S_{P,k}$ for $0 < k < OP$ are hyperboloids: for $k = 0$ the real points of $S_{P,0}$ span an R^{n-1}; for $k = OP$ the real points of $S_{P,OP}$ are the points of the line passing through O and P.

b. If X does not contain $n+1$ independent points, it follows that $X \subseteq R^{n-1}$ and the induction holds. Otherwise, let $O, P_1, \ldots, P_n$ be $n+1$ independent points of X. We fix a cartesian coordinate system with origin at O and the first $n-1$ coordinate axes in the R^{n-1} determined by $O, P_1, \ldots, P_{n-1}$.

c. From *a* it follows that the points of X are contained in the union (with respect to the k_i) of the intersection (with respect to i) of the quadrics S_{P_i,k_i} obtained from the pairs of points $OP_1, OP_2, \ldots, OP_n$. We can write

$$X \subseteq \bigcap_{i=1}^{n} \left(\bigcup_{k_i=1}^{OP_i-1} S_{P_i,k_i} \right) \cup \left\{ \begin{array}{l} \text{the points of } X \text{ that come from the intersections in} \\ \text{which a quadric is degenerate, that is, for } k_i = 0, \\ OP_i. \end{array} \right\}$$

If an S_{P_i,k_i} is degenerate, it is immediate that the intersection either consists of a finite set of points or is contained in an R^{n-1}, so that its contribution to the cardinality of X is a finite number of points.

d. We consider the real intersection of $n-1$ nondegenerate quadrics

$$\bigcap_{i=1}^{n-1} S_{P_i,k_i} \quad \text{with } k = (k_1, \ldots, k_{n-1}) \text{ fixed.}$$

This is either a finite set of points or else is a curve γ_k of order 2^n with real points and symmetric with respect to the hyperplane $x_n = 0$, since all the quadrics that appear in the intersection possess this symmetry.

e. We intersect the curve γ_k with a quadric S_{P_n,k_n} ($k_n = 1, \ldots, OP_n - 1$). This last quadric is certainly not symmetric with respect to the hyperplane $x_n = 0$.

f. If γ_k is irreducible, it cannot lie entirely on any S_{P_n,k_n} (the proof is analogous to the case $n = 3$). Hence, the real intersection is a finite set of points. If γ_k is reducible and $\gamma_k \cap S_{P_n,k_n}$ is not a finite set, then an irreducible component $\overline{\gamma}_k$ of the curve γ_k lies in S_{P_n,k_n}, and the order of $\overline{\gamma}_k$ is less than the order of γ_k.

g. If no point of X lies on $\overline{\gamma}_k$, the proof is finished. Otherwise, let P_{n+1} be a point of X lying on the curve $\overline{\gamma}_k$. In this case, it is required that the other points of X also be at integral distance from P_{n+1}, and, hence, that they lie in the intersection of $\overline{\gamma}_k$ with the $OP_{n+1} + 1$ quadrics $S_{P_{n+1}, k_{n+1}}(k_{n+1} = 0, 1, \ldots, OP_{n+1})$.

We can immediately exclude the case in which the quadrics are degenerate (see *c*). Now, either the real intersection $\overline{\gamma}_k \cap S_{P_{n+1}, k_{n+1}}$ is a finite set of points for every $k_{n+1} = 1, \ldots, OP_{n+1} - 1$, in which case the proof is already completed, or the real intersection of $\overline{\gamma}_k$ with a quadric is a curve whose order is lower than that of $\overline{\gamma}_k$.

By repeating the procedure outlined in *g*, we shall surely stop after a finite number of steps, because we find that the real intersection either is a finite set of points, or it contains no points of X, or it is a curve of order at most $n - 1$. In this last case it is known that the curve must lie in a subspace of dimension at most $n - 1$, and, hence, in particular, $X \subseteq R^{n-1}$.

4. We now give another demonstration of the result of Paragraph 1, which does not, however, give any idea of how the possible points must be distributed in the plane.

Given a triangle OAB in the plane, we fix a system of coordinates as in Paragraph 1. Let $OA = a$, $OB = b$, and $OC = c$, and let φ = angle AOB. We wish to find the points of the plane at integral distance from the vertices of the triangle. Let P be such a point, and set $OP = m$, $AP = m - k$, and $BP = m - h$, with m a positive integer and h, k integers. By the triangle inequality (see Paragraph 1), we have $|k| \leq a$, $|h| \leq b$, and, hence, if we denote the integral part of a by α and the integral part of b by β, we see that k can take only the values $0, \pm 1, \ldots, \pm\alpha$, and h only the values $0, \pm 1, \ldots, \pm\beta$. The coordinates x, y of P are solutions of the system of equations:

$$\begin{cases} x^2 + y^2 = m^2 \\ (x - a)^2 + y^2 = (m - k)^2 \\ (x - b\cos\varphi)^2 + (y - b\sin\varphi)^2 = (m - h)^2. \end{cases} \tag{2}$$

Substituting m^2 in place of $x^2 + y^2$ in each of the last two equations one finds

$$x = \frac{a^2 - k^2}{2a} + \frac{km}{a}$$

$$y = \frac{b^2 - h^2}{2b\sin\varphi} + \frac{hm}{b\sin\varphi} - \frac{\cos\varphi\, x}{\sin\varphi}$$

which shows that for every integral triple (k, h, m) there is a point (x, y). Now, the first equation of (2) gives a second-degree, nonidentical equation in m, whose coefficients are functions of h, k, a, b, $\cos\varphi$, $\sin\varphi$. (If a, b, and AB are integers, $\cos\varphi$ is rational and m is an integral solution of a diophantine equation.) Since as k and h vary one obtains $(2\alpha + 1)(2\beta + 1)$ such equations, we find at most $2(2\alpha + 1)(2\beta + 1)$ integral values for m and a like number of points at ID from the vertices of the triangle.

The generalization to R^n is analogous. Hence, we may state the following

Theorem: Given an $n + 1$-hedron in R^n with vertices $O, P_1, P_2, \ldots, P_n$, then there are at most

$$2\prod_{i=1}^{n}(2\alpha_i + 1)$$

points of R^n that have integral distance from the vertices of the $n + 1$-hedron; here α_i is the integral part of OP_i.

5. At this point, it is natural to ask if for any given integer n there is a configuration of n points of the plane that are mutually at ID. The answer is affirmative, and here we give a first method to construct such configurations.

Euler gave a construction (recorded, naturally, by Dickson) of polygons having sides, chords, and area all rational numbers, inscribed in a circle of radius $R = 1$: one selects n "Heron angles" $\alpha_1, \alpha_2, \ldots, \alpha_n$, that is, angles with rational sine and cosine, whose sum is less than π, and then, having fixed a point P_0 on the circumference of the circle with center at O and radius $R = 1$, one places P_1 on the circumference in a way such that $P_0OP_1 = 2\alpha_1$; then P_2 so that $P_1OP_2 = 2\alpha_2$; and so on. It is evident that the sides are $P_0P_1 = 2R \sin \alpha_1$, $\ldots, P_{i-1}P_i = 2R \sin \alpha_i, \ldots, P_nP_0 = 2R \sin(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$, that the chords are $P_iP_j = 2R \sin(\alpha_{i+1} + \cdots + \alpha_j)$ for $i < j$, and that the area is

$$A = (R/2)(P_0P_1 \cos \alpha_1 + P_1P_2 \cos \alpha_2 + \cdots + P_nP_0 \cos \alpha_{n+1});$$

here $\alpha_{n+1} = \pi - (\alpha_1 + \cdots + \alpha_n)$ is obviously a Heron angle. By the addition formula for the sine and the cosine, all sides and chords are rational numbers, and so is the area.

Set $t_i = \tan(\alpha_i/2) = p_i/q_i$ with p_i, q_i relatively prime integers. Then

$$\sin \alpha_i = 2p_iq_i/(p_i^2 + q_i^2) \quad \text{and} \quad \cos \alpha_i = (q_i^2 - p_i^2)/(p_i^2 + q_i^2).$$

Hence, it is clear that it suffices to take a circle with radius

$$4R = \prod_{i=1}^{n} (p_i^2 + q_i^2)$$

in order to obtain a similar polygon with sides and chords all integral numbers.

Let us see how it is possible to "improve" on the construction of Euler. Let $P_0P_1 \ldots P_n$ be a polygon, with rational sides and chords, inscribed in a circumference with center O and radius R, not necessarily rational. Set

$$P_{i-1}OP_i = 2\alpha_i \quad (i = 1, \ldots, n).$$

Since the angle $P_{i-1}P_{i+1}P_i = \alpha_i$, α_i is an angle of the rational-sided triangle $P_{i-1}P_iP_{i+1}$; hence, $\cos \alpha_i$ is rational, and also

$$\tan^2(\alpha_i/2) = (1 - \cos \alpha_i)/(1 + \cos \alpha_i)$$

is a rational number, for $i = 1, 2, \ldots, n$ (here $P_{n+1} = P_0$). Set

$$\tan(\alpha_i/2) = (p_i/q_i)d_i^{1/2}$$

with d_i a positive square-free integer and p_i, q_i integers for each i. Since $P_{i-1}P_i$ is rational, we must have $R = c_i d_i^{1/2}$ with c_i rational ($i = 1, \ldots, n$). But then $d_1 = d_2 = \cdots = d_n = d$. In conclusion, we must have

$$\tan(\alpha_i/2) = (p_i/q_i)d^{1/2} \quad \text{and} \quad R = cd^{1/2}$$

with c rational.

Conversely, consider a circle of radius $R = cd^{1/2}$, with c rational and d a square-free integer; it is then possible to inscribe in it a polygon $P_0 \ldots P_n$, for any given n, with rational sides and chords. To achieve this, just select angles α_i such that

$$\alpha_1 + \cdots + \alpha_n < \pi \quad \text{and} \quad \tan(\alpha_i/2) = (p_i/q_i)d^{1/2} \quad (p_i, q_i \text{ integers}),$$

and recall that

$$P_iP_j = 2R \sin(\alpha_{i+1} + \cdots + \alpha_j) \text{ for } i < j.$$

Hence, we have established the following theorem.

Theorem: A necessary and sufficient condition in order that a circle of radius R may circumscribe a polygon with rational sides and chords is that $R = cd^{1/2}$, with c rational and d a square-free positive integer. The area of the polygon is rational if and only if $d = 1$.

6. Another method ("the kite") for constructing a configuration of $2n + 3$ noncollinear points mutually at ID, with n fixed in advance, is the following. One selects n Pythagorean triples x_i, y_i, z_i, that is, integral solutions to the equation $x^2 + y^2 = z^2$. Let

$$a = \prod_{i=1}^{n} x_i, \quad b_j = y_j \prod_{i \neq j} x_i, \quad c_j = z_j \prod_{i \neq j} x_i.$$

For each i, we have $a^2 + b_i^2 = c_i^2$. Fix a point O in the plane, and let A be a point at distance a from O.

One can place n points $P_1, \ldots, P_n$ on the line through O perpendicular to OA, with P_i at distance b_i from O. Let A' be the point symmetric to A with respect to O and let $Q_1, \ldots, Q_n$ be the points symmetric to $P_1, \ldots, P_n$ (see Figure 2). The points $O, A, P_1, \ldots, P_n, A', Q_1, \ldots, Q_n$ are $2n + 3$ noncollinear points of the plane mutually at ID, and more precisely,

$$P_iP_j = Q_iQ_j = |b_i - b_j|,$$
$$AP_i = AQ_i = A'P_i = A'Q_i = c_i,$$
$$OP_i = OQ_i = b_i, \quad OA = OA' = a.$$

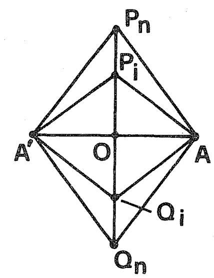

Fig. 2

Remark: It is not necessary that the angle P_iOA be a right angle. It must, however, be an angle φ with $\cos \varphi = p/q \in Q$. Then one has $\sin \varphi = d^{1/2}/q$ with d a positive integer. Let O, A, and P be as in Figure 3, a, b, c are integral solutions of the equation

$$x^2 + y^2 - 2xy \cos \varphi = z^2. \tag{4}$$

Set

$$\begin{cases} X = x \cos \varphi - y \\ Y = x \sin \varphi \\ Z = z \end{cases} \quad \text{so that} \quad \begin{cases} x = Y/\sin \varphi = Yq/d^{1/2} \\ y = Y \cos \varphi / \sin \varphi - X = Yp/d^{1/2} - X \\ z = Z \end{cases}$$

Equation (4) becomes

$$X^2 + Y^2 = Z^2, \tag{5}$$

and then

$$X = h^2 - k^2 d$$
$$Y = 2hkd^{1/2}$$
$$Z = h^2 + k^2 d$$

are the solutions of (5) as h and k range over Z (the ring of integers); hence,

$$x = 2hkq$$
$$y = 2hkp - (h^2 - k^2 d)$$
$$z = h^2 + k^2 d$$

are integral solutions of (4), as h and k range over Z. Having selected n solutions of (4), by picking n pairs (h, k), the kite method outlined at the beginning of this section supplies $n + 2$ noncollinear points mutually at ID (see Figure 3).

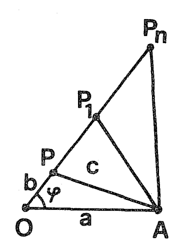

Fig. 3

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MEANS, CIRCLES, RIGHT TRIANGLES, AND THE FIBONACCI RATIO

ROBERT SCHOEN
University of Illinois, Urbana IL 61801

In looking for a convenient way to graph the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) of two positive numbers, I came across a connection between Kepler's "two great treasures" of geometry, the Pythagorean Theorem and the Golden Ratio, as well as several attractive geometric patterns.

Let us take a and b as the two positive numbers to be averaged and let

$$a + b = k. \tag{1}$$

The three means are defined as

$$\mathrm{AM}(a, b) = \frac{a+b}{2} = \frac{k}{2} \tag{2}$$

$$\mathrm{GM}(a, b) = \sqrt{ab} \tag{3}$$

$$\mathrm{HM}(a, b) = \frac{2ab}{a+b} = \frac{2ab}{k}. \tag{4}$$

To graph the three means, recall that a perpendicular line from a point on a circle to a diameter of the circle is the mean proportional (i.e., geometric mean) of the two segments of the diameter created by the line. In Figure 1, diameter AB, of length k, is composed of line segment $AD = a$ and line segment $DB = b$. The perpendicular DE is the geometric mean. When O is the center of the circle, the AM is equal to any radius, e.g., AO and OB. To find the harmonic mean, we proceed in the following manner. Construct a perpendicular to the diameter at the center O of height equal to DE, say line OP. Next, construct the perpendicular bisector of AP that meets diameter AB at C. Let Q be the center of a circle passing through A, B, and point C on AB. Since OP is the geometric mean of AO and OC, we have $OC = 2ab/k$, and thus the desired HM is line segment OC.

$\frac{AD}{AB} = \frac{3}{4}$

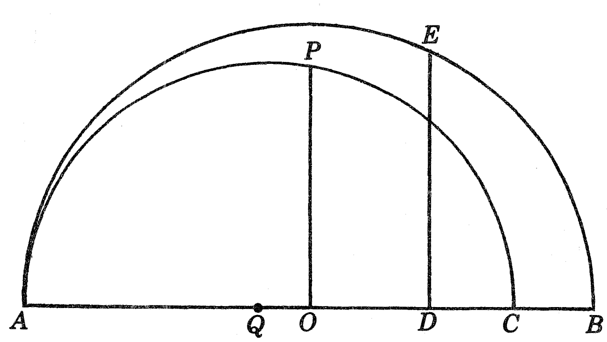

Fig. 1 Constructing the Arithmetic, Geometric, and Harmonic Means