

A NON-FIBONACCI SEARCH PLAN WITH FIBONACCI-LIKE RESULTS

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This article describes a nondeterministic search plan, hereinafter called the *mid-point technique*. While not optimal in the minimax sense, the plan offers several possible advantages over the Fibonacci technique. Further, the expected value of the reduction ratio at each stage is identical to the reduction ratio achieved by the minimax optimal Fibonacci method.

INTRODUCTION

Search techniques often use the minimax criterion as the assumed measure of effectiveness. As a result of Kiefer's pioneering work [4] demonstrating the minimax optimality of the Fibonacci search technique, a number of authors have focused attention on this particular search method. See, for instance, [1], [3], [5], [6], [7], and [8].

Unfortunately, in the authors' opinions, there are three disadvantages associated with the Fibonacci technique. First, the plan requires that the final reduction ratio be specified prior to beginning the search. Second, the Fibonacci search is one of the more complex unimodal sequential search techniques available, and this complexity may cause some potential users to avoid the Fibonacci technique in favor of a simpler method such as the dichotomous search or the golden section search [6]. Finally, if there is an upper bound on the number of experiments permitted, it may be impossible to achieve the required reduction ratio, i.e., the Fibonacci method does not provide the user with the option to gamble.

On a more fundamental level, the minimax criterion of optimality itself is open to challenge. The extremely pessimistic and jaundiced view of nature inherent within the minimax criterion may not represent a desirable framework from which to view the search procedure. While possibly valid for cases of warfare or for investors with extreme risk aversion, the minimax assumption of a malevolent opponent capable of altering the probabilities inherent within any gamble should be looked at with some skepticism. Murphy's Law notwithstanding, it is not reasonable to assume that all gambles taken by the searcher will necessarily be losing ones.

ASSUMPTIONS OF THE MID-POINT TECHNIQUE

The mid-point technique utilizes five assumptions. The first four are readily recognizable as being common ones often employed in search procedures. The fifth represents a significant departure from the minimax optimal Fibonacci method.

1. The response variable (y) is a function of the independent variable (x) and has a maximum (y^*) at $x = x^*$. The purpose of the search is to determine or approximate the value of x^* .

2. The function is unimodal; that is, given two experiments x_1 and x_2 with $x_1 < x_2$, let their outcomes be y_1 and y_2 , respectively. Then $x_2 < x^*$ implies $y_1 < y_2$ and $x_1 > x^*$ implies $y_1 > y_2$.

3. The minimum separation distance (ϵ) between experiments is negligible.

4. The original interval of uncertainty for x can be scaled to $[0, 1]$.

5. A priori, any given interval of finite length is assumed to have the same probability of containing x^* as any other interval of the same length (where both intervals lie within the remaining interval of uncertainty).

MECHANICS OF THE MID-POINT TECHNIQUE

The first experiment, x_1 , is placed at the center of the interval, and the second experiment, x_2 , is placed at ϵ (the minimum possible separation) to the right of x_1 . If $y_1 > y_2$, the interval $[x_2, 1]$ is dropped from further consideration. If $y_2 > y_1$, the interval $[0, x_1]$ is discarded. Under the assumptions of unimodality and negligible separation distance, this will necessarily reduce the interval of uncertainty to one-half of its original length. The third experiment, x_3 , is then placed at the center of the remaining interval. The third experiment will either halve the interval of uncertainty or reduce it by ϵ (the distance between x_1 and x_2). Under assumption 5, each of these mutually exclusive and exhaustive events is assumed to occur with probability 0.5. In the former case, the fourth experiment is again placed at the center of the remaining interval of uncertainty with the outcome of x_4 determining whether or not the remaining interval is significantly reduced. For the latter case (x_3 having negligible effect on the reduction ratio), x_4 is placed a distance of ϵ from x_3 , and the fourth experiment necessarily reduces the interval of uncertainty to one-half its previous length.

Figure 1 represents two of the six possible sets of experimental outcomes leading to a reduction ratio of 4.

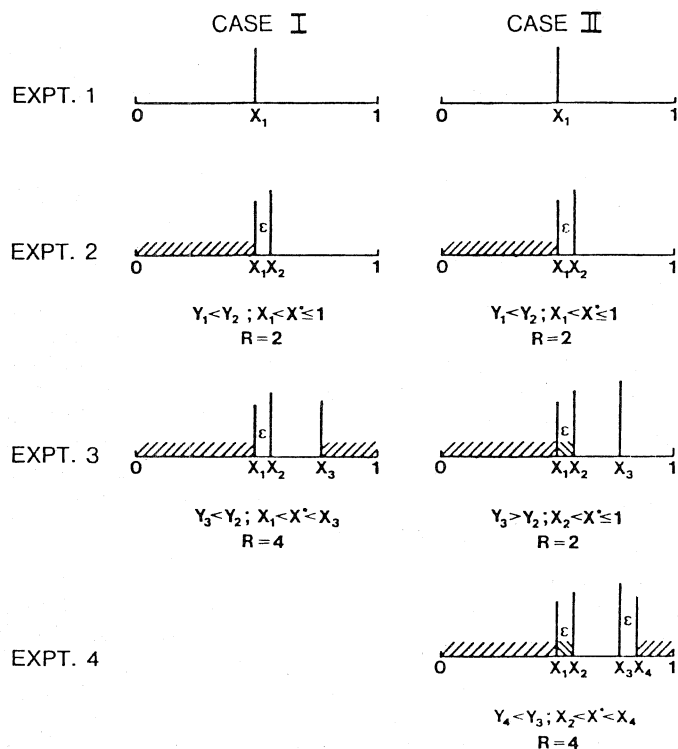


Fig. 1

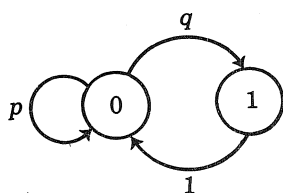
This search procedure continues until either a satisfactory reduction ratio has been attained or until the maximum number of experiments permitted has been run.

MODELING THE MID-POINT SEARCH TECHNIQUE

Each experiment of the mid-point technique results in exactly one of two possible outcomes: (1) the remaining interval of uncertainty is significantly reduced (by half), or (2) the interval of uncertainty is not significantly reduced. Clearly x_1 (by itself) has no effect, while the result of x_2 necessarily reduced the original interval by half. For $n \geq 3$, if x_n is placed in the center of the remaining interval, it will significantly reduce the interval (with probability 0.5) or it will fail to do so (also with probability 0.5). If, however, x_n is placed a distance ϵ from x_{n-1} , it will significantly reduce the interval (with probability 1) or will fail to do so (with probability 0).

It is thus natural to describe each experimental outcome as resulting in either a "success" (a significant reduction of the interval of uncertainty) or a "failure" (no significant reduction achieved). Further, the probabilities for achieving success or failure on each experimental trial depend exclusively on where the experiment is placed (either in the center of the remaining interval, or a distance ϵ from the last experiment), where placement depends upon the information derived from the previous experiment.

This suggests the use of a Markov chain to model the process (see Figure 2). Transition to State 0 represents a success in the terminology described above, while a transition to State 1 represents a failure.



$$P = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$$

$$\pi = \pi P = \left(\frac{1}{1+q}, \frac{q}{1+q} \right)$$

Fig. 2

The first two transitions of the chain are deterministic. The first experiment results in the occurrence of State 1 and the second experiment results in State 0 with a probability of 1. In this particular application, $p = q = 0.5$. States 0 and 1 form an irreducible recurrent set. The process of interest is the return times to State 0, which clearly forms a renewal process.

In terms of the mid-point technique, each transition of the Markov chain represents one experiment. The result of the first experiment necessarily results in a failure, that is, the chain making the transition to State 1 (with probability 1). Since the second experiment is placed at a distance ϵ from x_1 , the result of x_2 is necessarily a success, that is, the chain making the transition from State 1 to State 0 (also with probability 1). This first visit to State 0 is called the first renewal, and the first visit and all subsequent returns to State 0 result in a halving of the remaining interval of uncertainty. Equivalently, each time the chain undergoes a renewal, the reduction ratio is effectively doubled.

Obtaining the probability mass function for the number of renewals in a fixed number of transitions is a relatively straightforward matter (see Appendix). From this mass function, the exact probability for the number of renewals can be computed. If the random variable N_n represents the number of visits to State 0 after n transitions of the chain, then the reduction ratio R_n after n transitions (experiments) is simply expressed as $R_n = 2^{N_n}$. Since the probability mass function for N_n has been completely specified, this also specifies the mass function for the various values that the random variable R_n takes. From this, the expected value of R_n immediately follows.

Table 1 lists the expected value of the reduction ratio after n transitions or experiments, for values of n ranging from one to ten. Readers of this journal will immediately recognize the Fibonacci sequence.

Table 1

Number of Experiments	Number of Renewals	Equivalent Reduction Ratio	Probability of Occurrence	Expected Value of Reduction Ratio
0	0	1	1.0	1
1	0	1	1.0	1
2	1	2	1.0	2
3	1	2	0.5	3
	2	4	0.5	
4	2	4	0.75	5
	3	8	0.25	
5	2	4	0.25	8
	3	8	0.625	
	4	16	0.125	
6	3	8	0.500	13
	4	16	0.4375	
	5	32	0.0625	
7	3	8	0.125	21
	4	16	0.5625	
	5	32	0.28125	
	6	64	0.03125	
8	4	16	0.3125	34
	5	32	0.500	
	6	64	0.171875	
	7	128	0.015625	
9	4	16	0.0625	55
	5	32	0.4375	
	6	64	0.390625	
	7	128	0.1015625	
10	8	256	0.0078125	89
	5	32	0.1875	
	6	64	0.46875	
	7	128	0.28125	
	8	256	0.05859375	
	9	512	0.00390625	

ADVANTAGES OF THE MID-POINT TECHNIQUE

1. When a search point, x_i , falls sufficiently close to x^* , the subsequent experiments, x_{i+k} , will all be successes with consequent rapid convergence. As an extreme example, the case where x^* is located at the center of the original interval of uncertainty can be considered. In this case, x^* will lie in the interval $[x_1, x_2]$. For x_2 and all subsequent experiments, the interval of uncertainty will be halved.

2. In many situations involving a direct search, the marginal cost of additional experiments is constant, while the marginal value of information rapidly decreases with the time required to obtain the information. The expected profit of the search under these circumstances may be larger when using the mid-point

technique than when using the Fibonacci method. For example, if a reduction ratio of at least 30 is required and if the cost of placing experiments is \$10 per experiment while revenues are $[100 - 0.1n^2]$, the Fibonacci search requires 9 experiments and gives a profit of \$1.90. The mid-point technique has an expected profit of \$13.50 and requires 6 to 10 experiments.

3. If the desired reduction ratio must be accomplished within a specified number of search points, the Fibonacci search may be incapable of meeting the requirement. Under this circumstance, a rational choice is to gamble and the mid-point technique does allow gambling, although it does not insure a winning gamble.

4. The mid-point technique is easier to use than the Fibonacci search.

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REFERENCES

1. M. Avriel & D. J. Wilde. "Optimality Proof for the Symmetric Fibonacci Search Technique." *The Fibonacci Quarterly* 4 (1966):265-269.
2. E. Cinlar. *Introduction to Stochastic Processes*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1975. Pp. 283-292.
3. S. M. Johnson. "Best Exploration for Maximum Is Fibonacciian." The Rand Corporation, RM-1590, 1955.
4. J. Kiefer. "Sequential Minimax Search for a Minimum." *Proc. Amer. Math. Soc.* 4 (1953):502-506.
5. L. T. Oliver & D. J. Wilde. "Symmetric Sequential Minimax Search for an Optimum." *The Fibonacci Quarterly* 2 (1964):169-175.
6. D. J. Wilde. *Optimum Seeking Methods*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1964. Pp. 24-50.
7. D. J. Wilde & C. S. Beightler. *Foundations of Optimization*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1967. Pp. 236-242.
8. C. Witzgall. "Fibonacci Search with Arbitrary First Evaluation." *The Fibonacci Quarterly* 10 (1972):113-146.

APPENDIX

The first two transitions of the Markov chain are strictly deterministic. The chain goes to State 1 and then to State 0 (all with probability 1). Therefore, for the purposes of this analysis, we can ignore the first two transitions and take State 0 (our renewal state) as the initial state of the chain. Diagrammatically, the chain appears as in Figure 2.

Let $N(t)$ be the number of renewals in $[0, t]$ where t represents time, and let m be the number of transitions that occur in this interval. If each transition is assumed to require one time unit, then m is the integer part of t . We are interested in the probability distribution for the number of renewals in a finite number of transitions, i.e., $P\{N(t) = k\}$ for the various values of k . Counting the initial state of the chain as a renewal, the total number of renewals is clearly equal to one plus the number of returns to State 0.

Let $f(\cdot)$ be the probability mass function of inter-renewal times. Then

$$f(1) = p \quad \text{and} \quad f(2) = q.$$

Let $f^k(k+n)$ be the probability of obtaining the k th return, the $(k+1)$ th renewal, on the $(k+n)$ th transition for $n = 0, 1, \dots, k$. Note that f^k is the k -fold convolution of $f(\cdot)$. A little algebra quickly reveals that

$$f(k+n) = \binom{k}{n} p^{k-n} q^n \quad \text{for } n = 0, 1, \dots, k.$$

For purposes of algebraic simplicity, let $m = k + n$. Then

$$f^k(m) = \binom{k}{m-k} p^{2k-m} q^{m-k}; \quad m = k, k+1, \dots, 2k,$$

where $f^k(m)$ is the probability of obtaining the k th return on the m th transition. Similarly,

$$f^{k-1}(m) = \binom{k-1}{m-k+1} p^{2k-m-2} q^{m-k+1}; \quad m = k-1, k, \dots, 2k-2.$$

Let

$$F^k(t) = \sum_{m \leq t} f^k(m); \quad m = k, k+1, \dots, 2k.$$

Note that F^k is the probability of obtaining the k th return, the $(k+1)$ th renewal, at or prior to time t , where the maximum value of m is the largest integer less than or equal to t . It follows immediately that

$$F^k(t) = P\{N(t) \geq k+1\}.$$

Similarly,

$$F^{k-1}(t) = \sum_{m \leq t} f^{k-1}(m) = P\{N(t) \geq k\}.$$

With F^k and F^{k-1} completely specified as above, and using

$$P\{N(t) = k\} = F^{k-1}(t) - F^k(t)$$

(see [2, Ch. 9]), the distribution of $N(t)$ can be determined.

Algebraic manipulation and simplification results in the following:

$$P\{N(t) = k\} = F^{k-1}(t) - F^k(t)$$

$$= \begin{cases} 0; & t < k-1 \\ p^{k-1}; & k-1 \leq t < k \\ p^{k-1} + \sum_{m=k}^{[t]} p^{2k-m} q^{m-k} \left[p^{-2} q \binom{k-1}{m-k+1} - \binom{k}{m-k} \right]; & k \leq t < 2k-2 \\ 1 - \sum_{m=k}^{[t]} \binom{k}{m-k} p^{2k-m} q^{m-k}; & 2k-2 \leq t < 2k \\ 0; & t \geq 2k \end{cases}$$

where $[t]$ is the integer part of t .

A short computer program was written in FORTRAN to calculate these probabilities as well as the mean, variance, standard deviation, and skew for the number of renewals and its equivalent reduction ratio. The number of transitions was varied from one to twenty in increments of one. The program was compiled and executed under WATFIV and run on an AMDAHL 470/V6 computer in well under 0.5 seconds. Copies of this program are available on request.
