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A HISTORY OF THE FIBONACCI Q-MATRIX AND A HIGHER-DIMENSIONAL PROBLEM

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To the memory of Verner E. Hoggatt, Jr.

One of the most popular and recurrent recent methods for the study of the Fibonacci sequence is to define the so-called Fibonacci Q-matrix

(1) $$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

so that

(2) $$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where $F_{n+1} = F_n + F_{n-1}$, with $F_1 = 1$, $F_0 = 0$.

Theorems may then be cited from linear algebra so as to give speedy proofs of Fibonacci formulas. Write $|A|$ for the determinant of a matrix A. Then it is well known that $|AB| = |A| \cdot |B|$, and in general $|A^n| = |A|^n$. The Fibonacci Q-matrix method then gives at once the famous formula

(3) $$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

which was first given by Robert Simson in 1753. Formula (3) is the basis for the well-known geometrical paradox attributed to Lewis Carroll in which a unit of area mysteriously appears or disappears upon dissecting a suitable square and reassembling into a rectangle.

Where did this Q-matrix method originate? The object of the present paper is to give a tentative answer to this question, and present a reasonably complete bibliography of papers bearing on the use of such a matrix for the study of Fibonacci numbers. An unsolved problem is included.

The phrase "Q-matrix" seems to have originated in the master's thesis of Charles King [10]. At least, Basin and Hoggatt [16] cite this source, and from then on the idea caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in our *Fibonacci Quarterly* authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became a central tool in the analysis of Fibonacci properties. Vern Hoggatt carried on a far-ranging correspondence in which he jotted down ideas and made innumerable suggestions for further research. For example, his letters to me make up a foot-high stack of paper very nearly, representing creative thinking going on for 20 years. His contagious enthusiasm for research and the properties of numbers infected all whom he met or wrote to, and it seems to me that Vern must have been a major force for popularizing the Q-matrix method. Vern wrote me many letters, beginning in 1962, about using the Q-matrix method to study the Fibonacci polynomials and other related systems. He was very modest about claiming any credit for ideas and would often outline some method to me and then say "but this is probably pretty well known to you." Sometimes it was; more often not.

However, an early place that the Fibonacci matrix seems to appear in the form we know it is in an abstract by Joel Brenner [6], which I shall quote in detail for its historic significance:

"The n-th power of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is

$$\begin{pmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{pmatrix},$$

where u_n is Fibonacci's number. More generally, the n-th power of $\begin{pmatrix} a-b & -ab \\ 1 & 0 \end{pmatrix}$ is

$$\begin{pmatrix} u_{n+1} & -abu_n \\ u_n & -abu_{n-1} \end{pmatrix},$$

where $u_n = \dfrac{a^n - b^n}{a \quad b}$ is Lucas' number. From these facts it is easy to deduce a part of the general theory of these numbers.

"The sequences $u_n = A_1u_{n-1} + \cdots + A_ru_{n-r}$ have properties some of which are quickly obtained from the study of a matrix of dimension r which generalizes the matrices above."

In copying the abstract I have corrected several misprints. Vern and I used to discuss the history of the Q-matrix, and he published a 'belated acknowledgement' in our *Quarterly* [28] which appears as a note that was never listed in the volume index and thus has remained hard to locate. I shall quote the acknowledgement here in full:

"The first use of the Q-matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title 'Lucas' Matrix.' This abstract appeared in the March, 1951 *American Mathematical Monthly* on pages 221 and 222. The basic exploitation of the Q-matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title 'Some Further Properties of the Fibonacci Numbers.' Further utilization of the Q-matrix appears in the *Fibonacci Primer* sequence parts I-V."

To show that there was an active undercurrent of Fibonacci matrix activity around 1949-51, I wish next to quote an abstract by David DeVol [5] which appears, curiously, in the issue just preceding that in which Brenner's abstract turns up:

"Defining Fibonacci sequences by the property $u_{n+1} = u_n + u_{n-1}$, several relations between the terms are easily obtained by the manipulation of two-by-two matrices whose elements are terms of the sequence. The speaker concluded by pointing out a geometric connection between the Fibonacci sequences and the sequences of polygonal numbers."

Besides this there was a paper by J. Sutherland Frame [4] in 1949 that used matrices to study continued fractions, and the matrix

$$M_1 = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

appears, but no mention is ever made of the Fibonacci numbers per se. Matrix analysis of continued fractions is an old story also.

Rosenbaum [8] uses the matrix

$$R = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

to get an explicit formula for F_n, but does not consider R^n.

Miles [9] uses the matrix

$$A_n = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix},$$

but does not consider it as a power of a matrix.

Waddill's doctoral thesis [12] uses the matrix

$$\begin{pmatrix} 11 \\ 10 \end{pmatrix}$$

and also uses the third-order extensions. His later papers [26], [41] exploit the matrix further.

A *remarkable insight* is gained by examination of the well-known book of Schwerdtfeger [11]. On pages 104-105 he discusses Fibonacci polynomials and matrix methods due to Jacobsthal [1]. Schwerdtfeger uses a German gothic B for the matrix involved. Changing the lettering slightly we can summarize part of what Schwerdtfeger says as follows. Let

$$B = \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix}.$$

Then

$$B^n = \begin{pmatrix} f_n(b) & bf_{n-1}(b) \\ f_{n-1}(b) & bf_{n-2}(b) \end{pmatrix}, \tag{4}$$

where the Fibonacci polynomials are defined by $f_{n+1}(x) = f_n(x) + xf_{n-1}(x)$, with $f_0(x) = 1$ and $f_{-1}(x) = 0$. Explicitly

$$f_n(x) = \sum_{0 \le k \le n/2} \binom{n-k}{k} x^k.$$

Let

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad t = a + d = \text{trace} \neq 0.$$

Then there exists a matrix T such that $THT^{-1} = qB$. In fact

$$T = \begin{pmatrix} c & d \\ 0 & a+d \end{pmatrix}.$$

Finally

$$H^n = q^n T^{-1} B^n T, \tag{5}$$

where $b = -(ad - bc)/t^2$. This is an interesting result, since it shows how to express the n-th power of a 2 by 2 matrix in terms of powers of the "Q" matrix of a Fibonacci polynomial.

The only other reference I have noted in our *Quarterly* which cited Jacobsthal was the paper by Paul Byrd [14] in the very first issue of our journal. None cites Schwerdtfeger.

But the *concept* of a Fibonacci polynomial antedates Jacobsthal by a good many years. In fact, as Byrd [14] notes, a kind of Fibonacci polynomial was introduced as early as 1883 by E. Catalan, however, we shall not discuss this here. It is not entirely clear when in the pages of histroy a matrix was first used for such work.

Robinson [15] gives an extended discussion of matrix methods, citing many references, such as Bell [2], Ward [3], Brenner [7], and Rosenbaum [8]. He writes the matrix as

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and has

$$U^n = \begin{pmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{pmatrix}.$$

He calls U the *Fibonacci matrix*. Contrast this with Brenner who calls his matrix the *Lucas matrix*. I have not been able to ascertain whether Edouard Lucas himself used the matrix method.

Brennan [20] writes

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and higher-order extensions. But in [21] he writes

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and cites Basin and Hoggatt.

A novel application to group theory is afforded by the paper of White [22] who uses

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

to generate $GL(2, Z)$. See also Gale [27].

Bicknell finds the square root of the Q matrix [23], and goes on to fractional powers.

Lind [25] exhibits two matrices

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

such that $R^2 = S^3 = I$, hence R and S are of finite order. However,

$$RS = Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } (RS)^n = Q^n$$

which is easily seen never to equal I, so that RS is of infinite order. This does *not* happen in an abelian group, of course, where the product of two elements of finite order must again be of finite order.

Ivie [31] considers a general Q-matrix. He defines and uses the r by r matrix

$$Q_r = \begin{pmatrix} 1 & 1 & 0 & & \cdots & 0 \\ 1 & 0 & 1 & & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & & 0 & 0 \\ 1 & 0 & 0 & & \cdots & 0 \end{pmatrix},$$

which is well known as associated with higher-order linear recursions.

Serkland [33], in his master's thesis, uses a matrix analogous to Q in his study of the Pell sequence, which is therefore just a variant of the same consideration. See also a detailed report on this by Bicknell [37]. Here, of course,

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix},$$

where the Pell sequence is defined by

$$P_n = 2P_{n-1} + P_{n-2},\ P_1 = 1,\ P_2 = 2.$$

The Pell sequence is again studied by Ercolano [43].

Hoggatt and Bicknell-Johnson [42] use what have been called Morgan-Voyce polynomials b_n, B_n and they find the following. Let

$$A = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$(6) \qquad (AB)^n = \begin{pmatrix} b_n(xy) & xB_{n-1}(xy) \\ yB_{n-1}(xy) & b_{n-1}(xy) \end{pmatrix}.$$

Pollin and Schoenberg [45] turn the Q-matrix upside down in the form

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and use A^n in their study of the converse of the congruence p = prime implies $L_p \equiv 1 \pmod{p}$, where L_p is the Lucas number.

Our bibliography does not summarize all of the literature, but does give a good idea of what has been done with the Q-matrix and its extensions.

Now we wish to close with some remarks about problems that remain unsolved. These problems involve higher-dimensional determinants and matrices.

In my paper [19], I studied an operator I called a Turán operator, defined by

$$(7) \qquad Tf = T_x f(x) = T_{x,a,b} f(x) = f(x+a)f(x+b) - f(x)f(x+a+b).$$

It is easy to show that

$$(8) \qquad T_x \sin x = T_x \cos x = \sin a \sin b,$$

and, as an extension of (3), it is possible to prove that

$$(9) \qquad T_n F_n = F_{n+a}F_{n+b} - F_n F_{n+a+b} = (-1)^n F_a F_b,$$

so that (3) occurs when $a = 1$ and $b = -1$.

My paper obtained extensions of this formula by exploring some possible *extensions of determinants to three and four dimensions*. Thus it was found that

$$F_{n+a}F_{n+b}F_{n+c} - F_nF_{n+a}F_{n+b+c} + F_nF_{n+b}F_{n+a+c} - F_nF_{n+c}F_{n+a+b} = (-1)^n(F_aF_bF_{n+c} - F_cF_aF_{n+b} + F_bF_cF_{n+a}), \tag{10}$$

with further reductions, and yet the trouble is that there is *no unique* way to go about defining higher-dimensional determinants.

Since it is possible to prove (9) by means of skillful manipulations with a two-dimensional Q-matrix, one naturally desires to extend the idea to (10) and related formulas using a three-dimensional Q-matrix. *Again, there seems to be difficulty in defining three-dimensional matrices.* It would be necessary to see how to extend the property mentioned at the outset of this paper,

$$|A \cdot B| = |A| \cdot |B| \tag{11}$$

for square two-dimensional matrices. *How can this be extended, if indeed at all, to three-dimensional matrices?* We leave this unsolved problem for the reader.

If Vern Hoggatt had worked on this problem we might have a solution already. Such was the enthusiasm he had for the Q-matrix, but he never got around to exploring this higher-dimensional direction.

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SOME DIVISIBILITY PROPERTIES OF PASCAL'S TRIANGLE

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This paper is dedicated to the memory of Professor V. E. Hoggatt, Jr., whose happy enthusiasm for mathematics has been an inspiration to all who knew him and whose friendship has enormously enriched the lives of so many, including, in particular, the present author.

1. INTRODUCTION

Let p denote a prime and let m, n, h, k, and α denote integers with

$$0 \le k \le n,\ 1 \le h \le n,\ m \ge 1, \text{ and } \alpha \ge 1.$$

Let $\Delta_{n,k}$ denote the triangle of entries

$$\begin{array}{ccc} & \binom{nm}{km} & \\ & \cdot^{\cdot^{\cdot}} \quad \cdot_{\cdot_{\cdot}} & \\ \binom{nm+m-1}{km} & \cdots & \binom{nm+m-1}{km+m-1} \end{array}$$

from Pascal's triangle. And let $\nabla_{n,h}$ denote the triangle of entries from Pascal's triangle indicated by

$$\begin{array}{ccc} \binom{nm}{hm-m+1} & \cdots & \binom{nm}{hm-1} \\ & \cdot_{\cdot_{\cdot}} \quad \cdot^{\cdot^{\cdot}} & \\ & \binom{nm+m-2}{hm-1} & \end{array}$$

For $m = p^{\alpha}$, we showed in [2] that all elements of Pascal's triangle not contained in some $\Delta_{n,k}$ (i.e., those contained in some $\nabla_{n,h}$) are congruent to 0 modulo p, that, modulo p, there are precisely p distinct triangles $\Delta_{n,k}$, and that these triangles can be put in one-to-one correspondence with the residues 0, 1, 2, ..., $p - 1$ in such a way that the triangle of triangles