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PROPORTIONAL ALLOCATION IN INTEGERS

A. P. HILLMAN
University of New Mexico, Albuquerque NM 87131

Dedicated to the memory of Vern Hoggatt

The U.S. Constitution mandates that "Representatives shall be apportioned among the several states according to their respective numbers. . . . The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative." Implementation is left to Congress.

Controversy arose over the first reapportionment. Congress passed a bill based on a method supported by Alexander Hamilton. President George Washington used his first veto to quash this bill, and an apportionment using Thomas Jefferson's method of "greatest divisors" was adopted. This matter is still controversial. Analyses, reviews of the history, and proposed solutions are contained in the papers [3], [4], and [5] in the *American Mathematical Monthly*.

The purpose of this paper is to cast new light on various methods of proportional allocation in natural numbers by moving away from the application to reapportionment of the House of Representatives after a census and instead considering the application to division of delegate positions among presidential candidates based on a primary in some district.

1. THE MATHEMATICAL PROBLEM

Let $N = \{0, 1, 2, \ldots\}$ and let W consist of all vectors $V = (v_1, \ldots, v_n)$ with components v_i in N and dimension $n \geq 2$. Let the *size* of such a V be

$$|V| = v_1 + \cdots + v_n.$$

An allocation method is a function F from $N \times W$ into W such that

$$F(s, V) = S = (s_1, \ldots, s_n) \text{ with } |S| = s.$$

We will sometimes also write $F(s, V)$ as $F(s; v_1, \ldots, v_n)$. $S = F(s, V)$ should be the vector in W with size s and the same dimension as V which in some sense is most nearly proportional to V.

A property common to all methods discussed below is the *fairness* property that

$$s_i \geq s_j \text{ whenever } v_i > v_j. \tag{1}$$

Note that $s_i > s_j$ can occur with $v_i = v_j$ since the requirement that each s_i be an integer may necessitate use of tie-breaking (e.g., when all v_i are equal and s/n is not an integer).

2. TYPES OF APPLICATIONS

For reapportionment of the U.S. House of Representatives, at present $n = 50$, $s = 435$, the v_i are the populations of the states (say in the 1980 census), and s_i is the number of seats in the House to be alloted by the method to the ith state. Proportional allocation could also be used to divide congressional committee positions among the parties or to allot Faculty Senate positions to the various colleges of a university.

We want to get away from the relatively fixed nature of the dimension n and the constitutional requirement that each $s_i \geq 1$ in the reapportionment of the House problem and therefore, in the main, will use language and examples appropriate for the application to presidential primaries.

3. TWO EXTREME METHODS

The "plurality takes all" method P has

$$P(s;\ v_1,\ \ldots,\ v_n) = (s_1,\ \ldots,\ s_n)$$

with $s_k = s$ if v_k is the largest of the v_i and $s_i = 0$ for all other i. This method is used in elections in which $s = 1$, e.g., elections for mayor or governor. It is also used in allocating the total electoral vote of a state based on the vote for president in general elections. This method is certainly not one of "proportional" allocation.

Perhaps at the other extreme is the "leveling" method

$$L(s;\ v_1,\ \ldots,\ v_n) = (s_1,\ \ldots,\ s_n)$$

in which the s_i are as nearly equal as possible. That is, if $s = qn + r$ with q and r integers such that $0 \leq r < n$, then $s_i = q + 1$ for the r values of i with the largest components v_i and $s_i = q$ for the other values of i. This is the method used to allocate the 100 seats in the U.S. Senate among the 50 states. It too is not a method of proportional allocation.

4. ONE PERSON, ONE EFFECTIVE VOTE

We find it helpful to preface our discussion of proportional allocation with the consideration of a proportional representation election to choose people for a city council, or a school board, or to represent the electorate in some other way. As a means of achieving proportional representation it is decided to give each voter only one vote; the s candidates with the highest votes will be declared elected.

Each voter has a favorite candidate but a vote for the favorite may be a wasted vote because that candidate is so strong as not to need the vote in order to be elected, or is too weak to be in contention. If enough electors change their votes in fear of such wastage, the results may be a serious distortion of their wishes and may involve an even greater wasting of votes.

But there are methods which provide near optimum effectiveness for the total vote. They involve a preferential ballot on which each voter places the number 1 next to the voter's first choice, 2 next to the second choice, etc. Then, a very sophisticated system is used to transfer a vote when necessary to the highest indicated choice who has not yet been declared elected or been eliminated due to lack of support. Such a method is used to select members of the Nominating Committee of the American Mathematical Society (see [6]) and to elect members of the Irish Parliament (see [2]).

The following arithmetical question arises in such single vote, multiposition elections: If there are v voters and s positions to be filled, what is the smallest integer q such that q votes counted for a candidate will guarantee election under all possibilities for the other ballots? Clearly, the answer is the smallest q such that $(s + 1)q > v$ or

$$q = [(v + 1)/(s + 1)], \text{ where } [x] \text{ is the greatest integer in } x. \quad (2)$$

If Americans in general were more educated politically and mathematically, such a method might be used to elect delegates to presidential nominating conventions. Then an elector could number choices based on whatever criteria were considered most important, such as the presidential candidate backed, major issues, or confidence in a specific candidate for delegate.

At best, our primaries allow electors to express one choice for president. What "one person, one vote" mechanism could we use to assign each vote to a candidate for delegate to make best use of the only information we have, that is, the number v_i of votes for presidential candidate C_i? If the v_i people voting for C_i knew that they could maximize the number of delegates allocated to C_i by dividing into s_i equal-sized subsets with each subset voting for a different delegate candidate pledged to C_i, the only information we have indicates that they would do so. The result would be the allocation of the Jefferson method, which we discuss in the next section.

5. JEFFERSON'S GREATEST DIVISOR METHOD

In a given primary, let there be n presidential candidates C_i, let v_i be the number of votes for C_i, and let s be the total number of delegate seats (in a given political party) at stake in that primary. Suppose that the presidential candidates have submitted disjoint lists of preferred candidates for delegate positions with the names on each list ranked in order of preference.

Now let us fix i and consider each individual vote for C_i as a single transferable vote which is to be assigned to one of the delegate candidates on the C_i list, with the assignment process designed to maximize the number s_i of people on this list winning delegate seats. Below we show inductively that the following algorithm performs this optimal assignment and determines all the s_i.

From the ns ordered pairs (i, j) with $1 \leq i \leq n$ and $1 \leq j \leq s$, choose the s ordered pairs for which v_i/j is largest. This may require a tie-breaking scheme (as is true of all allocation methods). Then the allocation s_a to candidate C_a is the number of (i, j) with $i = a$ among these s chosen pairs.

6. FIRST EXAMPLE

Here let $n = 4$ and the votes for four presidential candidates C_i in a given primary be given by the vector

$$(v_1, v_2, v_3, v_4) = V = (3110, 2630, 2620, 1640).$$

The necessary calculations and ordering for the Jefferson method J of allocating a total of s delegate positions among the four contending campaign organizations is shown for $1 \leq s \leq 22$ in the following table:

		C_1		C_2		C_3		C_4
v = vote received	(1)	3110	(2)	2630	(3)	2620	(4)	1640
$v/2$	(5)	1550	(6)	1315	(7)	1310	(11)	820
$v/3$	(8)	1036+	(9)	876+	(10)	873+	(16)	546+
$v/4$	(12)	777+	(13)	657+	(14)	655		410
$v/5$	(15)	622	(17)	526	(18)	524		
$v/6$	(19)	518+	(21)	438+	(22)	436+		
$v/7$	(20)	444+						
$v/8$		388+						

The position indicators in parentheses before various quotients v_i/j show the positions of these quotients when all quotients for all candidates are merged together in decreasing order. The number s_i of delegate seats to be alloted to presidential candidate C_i is the number of position indicators from the range 1, 2, ..., s which appear in the column for C_i. For example, when $s = 20$, the allotment to C_2 is $s_2 = 5$ since the five position numbers 2, 6, 9, 13, 17 from the range 1, 2, ..., 20 appear prior to quotients v_2/j in the C_2 column. The complete allocations for s = 20, 21, and 22 are

$$J(20, V) = (7, 5, 5, 3),$$
$$J(21, V) = (7, 6, 5, 3),$$
$$J(22, V) = (7, 6, 6, 3).$$

This example helps us in discussing the rationale for the method J. Let the total number of delegate seats to be alloted be 21. Think of 20 of the 21 spots as having already been alloted with the distribution

$$J(20, V) = (7, 5, 5, 3)$$

and ask to whom the 21st spot should be given. Clearly, C_3 is not entitled to a 6th spot before C_2 obtains a 6th spot. To see if C_1 should get an 8th, or C_2 a 6th, or C_4 a 4th, one looks at the largest quotient among $v_1/8$, $v_2/6$, and $v_4/4$. The 21st spot goes to C_2 on this basis. The result is the same as what would happen if we considered each vote for C_2 as a single transferable ballot which should be assigned so as to maximize the number of delegates pledged to C_2. Then the 2630 votes for C_2 could be assigned in six batches of at least 438 for delegate candidates pledged to C_2 and it would be impossible to assign the votes for the other presidential candidates in batches of at least 438 to more than 7 people pledged to C_1, 5 to C_3, and 3 to C_4.

Now, let us alter the above example by introducing new presidential candidates C_5, C_6, ..., C_n (some of whom may be mythical write-in names) with votes v_5, ..., v_n. We keep the total number of spots at $s = 20$, and note that the 20th largest v_i/j among the original candidates C_1, C_2, C_3, C_4 is 444+. Hence the allocation will be (7, 5, 5, 3, 0, 0, ..., 0) unless some new v_i is at least 445. Thus the method J has a built-in mechanism for distinguishing "real candidates" from "ego-trippers" and recipients of small batches of write-in votes deliberately wasted as a form of protest.

7. HAMILTON'S ROUNDING METHOD

Let us continue to use the votes vector v = (3110, 2630, 2620, 1640) of our example above. Let the number s of delegate positions available be 20. Hamilton's reasoning was similar to the following:

The "ideal" allocation of 20 positions in exact proportion to the v_i, but dropping the requirement of allocating in whole numbers, would be

C_1	C_2	C_3	C_4
6.22	5.26	5.24	3.28

If we have to change these to whole numbers, then clearly C_1 is entitled to at least 6, C_2 and C_3 are entitled to at least 5, and C_4 to at least 3. That disposes of 19 of the 20 positions. Who should get the 20th? Hamilton's method H, also called the Vinton Method, would give it to C_4 on the ground that his "ideal" allotment has the largest fractional remainder. (In Europe, this method is called the "greatest remainders" method.)

We note that the Hamilton allotment $H(20, V) = (6, 5, 5, 4)$, while the Jefferson allotment is $J(20, V) = (7, 5, 5, 3)$. Before we decide on which is "more nearly proportional," let us use the same votes vector V = (3110, 2630, 2620,

1640) but change from $s = 20$ to $s = 22$. Then the "ideal" decimal allocation becomes

C_1	C_2	C_3	C_4
6.842	5.786	5.764	3.608

and the Hamilton allotment $H(22, V)$ is (7, 6, 6, 3). The surprise is that adding two new positions, while keeping the votes the same, results in C_4 *losing* one spot and each of the others gaining one. This phenomenon using the method H is called the "Alabama Paradox" and was first noticed when the Census Office chief clerk, C. W. Seaton, showed that the apportionment under H of seats to Alabama in the House after the 1880 census would decrease from 8 to 7 if the House size were increased from 299 to 300 (with the same population figures).

Balinski and Young [3, p. 705] quote Seaton, after discovering the paradox, as writing that "Such a result as this is to me conclusive proof that the process employed in obtaining it is defective. . . . [The] result of my study of this question is the strong conviction that an entirely different process should be employed" and also quote [3, p. 704] Representative John C. Ball of Colorado as saying that "This atrocity which [mathematicians] have elected to call a 'paradox' . . . this freak [which] presents a mathematical impossibility."

Since Seaton's observation, Hamilton's method has not been used for reapportionment of the House. However, it is perhaps the most widely used method in elections. The 1980 delegate selection rules of one of our major political parties required that "this atrocity" be used.

Since the Jefferson method J allocates spots one by one as s increases, it is trivial to show that the Alabama Paradox cannot occur under J. (No parenthetic position number is erased when s increases by one.)

8. OTHER QUOTA METHODS

The discovery of the Alabama Paradox inspired a number of mathematicians to seek quota methods which are "house monotone," that is, quota methods which do not allow this particular type of paradox. These variations on H maintain the insistence that the "ideal" decimal allotments can be changed only through rounding up or down but they use other criteria than size of the decimal remainder to decide on which way to round. Such "quota" methods are described and justified in [3], [4], and [5]. One should note that these papers deal only with the application to reapportionment of the House. For this application, the mathematicians of the National Academy of Sciences are available and could use sophisticated mathematics such as that of [5].

Our contention is that, at least in applications to primaries, all quota methods exhibit other anomalies, and that the criticisms of Jefferson's method are not very relevant. For additional ammunition to bolster these assertions, we consider new examples.

9. A NEW PARADOX

For the remaining examples, we fix the number of delegate spots at $s = 20$ and vary the number n of presidential candidates. Using the same votes vector $V = (3110, 2630, 2620, 1640)$ as above, one finds that a sophisticated quota method Q, such as those in [3], [4], and [5], has in effect been forced to agree with the allotment (7, 5, 5, 3) of the Jefferson method to avoid the Alabama Paradox. Now we introduce five new presidential candidates $C_5, \ldots, C_9$ with C_9 a write-in candidate (my favorite is Kermit the Frog). Let the new votes vector be $V = (3110, 2630, 2620, 1640, 99, 97, 86, 84, 1)$.

Under the Jefferson method, the 20th quotient remains at 444+. Hence those v_i which are less than 445 do not influence the results and the allocation is

(7, 5, 5, 3, 0, 0, 0, 0, 0). However, no quota method Q can give the same result. The reason is that the one vote "wasted" on C_9 has reduced the "ideal" decimal allotment for C_1 below 6 and, thus, all quota (i.e., rounding) methods bar C_1 from having more than 6 delegates. This means that, under Q, one of the 367 people who voted for $C_5, \ldots, C_9$ took a delegate spot away from C_1 and gave it to C_2 or C_4. In this example, the write-in for C_9 is the vote that forced this anomaly.

The present author feels that such an effect is also "an atrocity" and is still paradoxical. Jefferson's method avoids this anomaly since under J a vote can take a spot away from C_i only by adding a spot for the candidate C_j for whom the vote was cast.

Altering Q, as long as it remains a quota method, can only make us change our example. No quota method is immune to this anomaly.

10. INTERNAL CONSISTENCY

Let F be an allocation method, $V = (v_1, \ldots, v_n)$, and

$$S = F(s, V) = (s_1, \ldots, s_n).$$

Let A be any proper subset of $\{1, 2, \ldots, n\}$, s' be the sum of the s_i for i in A, and V' and S' be the vectors resulting from the deletions of the components v_j of V and s_j of S, respectively, for all j not in A. If under all such situations we have $F(s', V') = S'$, we say that the method F is *internally consistent.* The discussion in the previous section indicates why no quota method can be internally consistent.

The Jefferson method J is easily seen to be internally consistent. So is the Huntington "Method of Equal Proportions," which is the one used in recent reapportionments of the House. This method E is the variation on J in which the quotients v_i/j are replaced by the functions $v_i/\sqrt{j(j-1)}$. Note that this function is infinite for $j = 1$ and is finite for $j > 1$. Hence in the application to apportionment of the House, one could interpret E as requiring that each state must be given one seat in the House before any state can receive two seats. Since this is required by the U.S. Constitution, E is a method that has this mandated bias toward states with very small populations and gradually decreases this bias as the population grows. References [3], [4], and [5] take the position that an acceptable apportionment method must be a quota method; they therefore reject J and E and all methods which we call internally consistent. Neither of these references mentions the fact that E "naturally" satisfies the constitutional requirement that each state must have at least one Representative. Despite this naturalness in using E for apportionment of the House, it seems to be an absurdity to use E in a presidential preference primary since single write-ins for enough names to make $n \geq s$ would force all allocations s_i to be in $\{0, 1\}$.

Balinski and Young [3, p. 709] ask: "Why choose one stability criterion rather than another? Why one rank-index than another? Why one divisor criterion than another." Later on the same page they quote a Feb. 7, 1929, report of the National Academy of Sciences "signed by lions of the mathematical community, G. A. Bliss, E. W. Brown, L. P. Eisenhart, and Raymond Pearl" as containing the statement that "Mathematically there is no reason for choosing between them." The word "them" refers to a number of methods which are internally consistent.

In the application to presidential primaries, one reason for choosing J over other methods if that J achieves the same results as the "single transferable ballot" method if one considers each vote for a presidential candidate C_i to be a ballot marked with perfect strategy solely for delegate candidates pledged to C.

11. DIVIDE AND CONQUER

The Hamilton method (and other quota methods) may allow a group to round an "ideal" allotment of 4.2 into 7 by the group breaking up into seven equal subgroups, each with an "ideal" allotment of 0.6. Thus, quota methods can reward fragmentation and seem especially inappropriate in selecting just one person to lead a political party (and perhaps the nation). Under Jefferson's method, no group can gain by dividing into subsets and no collection of groups can lose by uniting into one larger group.

When J was originally proposed for reapportionment, it was criticized for not being biased toward small states. The criticism by mathematicians, such as in [3], [4], and [5], is that it is not a quota (i.e., rounding) method.

12. UNDERLYING CAUSES OF ANOMALIES

Why does the Hamilton method allow the "Alabama Paradox" and why are the other, more sophisticated quota methods subject to regarding a vote for Z as a vote for Y and/or a vote against X? Basically, the trouble with all quota methods is that they mix the multiplicative operation with addition and subtraction. For example, they allow 0.1 to be rounded up to 1 but do not allow 8.99 to be "rounded" to 10. Thus, they allow the actual allotment to be ten times the "ideal" for one candidate while not allowing it to be 1.2 times the "ideal" for another. The characteristic feature of quota methods is the insistence that there be no integer strictly between the actual and the "ideal" allotment. Thus, there is a bound of 1 on this difference, although there is no bound on the corresponding ratio. A method that claims to give "most proportional" results should give more importance to the ratio than to the difference.

This author also feels that quota methods (for primaries) are wrong in insisting that v_i be at least $|V|/s$ to guarantee at least one delegate for C_i. The discussion of "single transferable ballot" methods (Section 4 above) indicates that this should be $[(|V| + 1)/(s + 1)]$ instead of $|V|/s$. Also, quota methods ignore the fact that many votes in a primary may unavoidably be just wasted votes. Using these wasted votes to determine "ideal" allotments allows a vote cast for Z to have the effect of a vote against X and/or a vote for Y. A minimal step in the right direction would be to delete the v_i for candidates who receive zero allocations from the total vote size $|V|$ in determining the "ideal" allotment. (This might entail iteration of some process.)

13. THE REAL LIFE EXPERIENCE

The 1980 delegate selection rules of one of our major parties for the national presidential nominating convention required that the "paradoxical atrocity" H be used. However, the paradoxes illustrated above could not occur because these rules also stated that candidates who received less than 15 percent of the vote in some primary were not eligible for delegate allocations. In reaction to the "plurality takes all" procedures of previous years, these rules also said that no candidate who received less than 90 percent of the total vote could be alloted all the delegate positions at stake in a given primary. If there were three candidates and their percentages of the total vote were 75, 14, and 11 percent, then any allocation under this patched up version of H would contradict some provision of these rules. So patches were added onto the patches described above. Contradictions were being discovered and patches added until all the delegates were selected and the issue became moot.

The Jefferson method is much simpler to use and would have achieved more or less the same overall result. At least one state recognizes the Jefferson method in its presidential primary act.

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IRRATIONAL SEQUENCE-GENERATED FACTORS OF INTEGERS

A. F. HORADAM

University of New England, Armidale, NSW, 2351

and

A. G. SHANNON

N.S.W. Institute of Technology, Broadway, NSW, 2007

Dedicated in respectful and affectionate remembrance to the memory of our good friend, Vern Hoggatt, a man and a mathematician of high quality.

1. INTRODUCTION

In Horadam, Loh, and Shannon [5], a generalized Fibonacci-type sequence $\{A_n(x)\}$ was defined by

$$(1.1)\qquad \begin{cases} A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1, \text{ and} \\ A_n(x) = xA_{n-2}(x) - A_{n-4}(x) \qquad (n \geq 4). \end{cases}$$

The notion of a proper divisor was there extended as follows:

Definition: For any sequence $\{U_n\}$, $n \geq 1$, where $U_n \in \mathbb{Z}$ or $U_n(x) \in \mathbb{Z}(x)$, the *proper divisor* w_n is the quantity implicitly defined, for $n \geq 1$, by $w_1 = U_1$ and $w_n = \max\{d\colon d|U_n,\ \text{g.c.d.}\ (d, w_m) = 1 \text{ for every } m < n\}$.

It was then shown that

$$(1.2)\qquad A_n(x) = \prod_{d|n} w_d(x)$$

and

$$(1.3)\qquad w_n(x) = \prod_{d|n} (A_d(x))^{\mu(n/d)}$$

where $\mu(n/d)$ are Möbius functions.