The Jefferson method is much simpler to use and would have achieved more or less the same overall result. At least one state recognizes the Jefferson method in its presidential primary act.

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IRRATIONAL SEQUENCE-GENERATED FACTORS OF INTEGERS

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Dedicated in respectful and affectionate remembrance to the memory of our good friend, Vern Hoggatt, a man and a mathematician of high quality.

1. INTRODUCTION

In Horadam, Loh, and Shannon [5], a generalized Fibonacci-type sequence $\{A_n(x)\}$ was defined by

$$(1.1)\qquad \begin{cases} A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1, \text{ and} \\ A_n(x) = xA_{n-2}(x) - A_{n-4}(x) \qquad (n \geq 4). \end{cases}$$

The notion of a proper divisor was there extended as follows:

Definition: For any sequence $\{U_n\}$, $n \geq 1$, where $U_n \in \mathbb{Z}$ or $U_n(x) \in \mathbb{Z}(x)$, the *proper divisor* w_n is the quantity implicitly defined, for $n \geq 1$, by $w_1 = U_1$ and $w_n = \max\{d\colon d|U_n,\ \text{g.c.d.}\ (d, w_m) = 1 \text{ for every } m < n\}$.

It was then shown that

$$(1.2)\qquad A_n(x) = \prod_{d|n} w_d(x)$$

and

$$(1.3)\qquad w_n(x) = \prod_{d|n} (A_d(x))^{\mu(n/d)}$$

where $\mu(n/d)$ are Möbius functions.

Elsewhere [8], Shannon, Horadam, and Loh have proved (with n replaced by $2n$) that

$$A_{4n}(x) = \sum_{j=0}^{[n-\frac{1}{2}]} (-1)^j \binom{2n-j-1}{j} x^{2n-2j-1}. \tag{1.4}$$

The background to this paper is that the authors were shown (Wilson [9], [10]) several numerical results relating to the sets of numbers in Table 2, and asked to establish a theoretical basis for these results. In the process, some useful further properties of (1.4) were developed.

A particular aim of this investigation is to use the generalized Fibonacci-type sequence to show that any integer $n > 0$ can be expressed as the product of (mostly) irrational numbers in an infinite number of ways according to a specific pattern.

Besides expressing our appreciation of the stimulation provided by Wilson ([9], [10]), we wish to register our thanks to A. Hartman and R. B. Eggleton [4] for their valuable comments, and to Professor G. E. Andrews, University of Pennsylvania, for the Hancock reference [3].

2. FACTORS, PROPER DIVISORS, AND TRIGONOMETRY

From (1.4) we observe that

$$\text{deg.}\left(\frac{A_{4n}(x)}{x}\right) = 2n - 2 \tag{2.1}$$

so that

$$\frac{A_{4n}(n)}{x} = 0 \tag{2.2}$$

has $n - 1$ squares of roots

$$\alpha_1^2, \alpha_2^2, \ldots, \alpha_{n-1}^2.$$

For notational convenience write

$$\beta_i = \alpha_i^2 \qquad i = 1, 2, \ldots, n-1. \tag{2.3}$$

Since the constant term in (2.2) is $(-1)^{n-1}n$, we have, from the theory of equations, that

$$n = \prod_{i=1}^{n-1} \beta_i \tag{2.4}$$

and also, with $j = 1$ in the left-hand side of (2.2) that

$$2n - 2 = \sum_{i=1}^{n-1} \beta_i. \tag{2.5}$$

Thus, to find the factors of any integer n, we seek the $n - 1$ β_i of (2.2), which by (1.2) can be obtained from the proper divisors of $A_{4n}(x)/x$. The first few of the $A_{4n}(x)/x$ are listed in Table 1 along with their factors and proper divisors.

For example, from Table 1, [5], and (1.2), $A_{20}(x)$ has as its factors

$$w_{20}(x) = x^4 - 5x^2 + 5,\ w_{10}(x) = x^2 - x - 1,\ w_5(x) = x^2 + x - 1,$$
$$w_4(x) = 1,\ w_2(x) = 1, \text{ and } w_1(x) = 1$$

trivially.

In the search for proper divisors, the (provable) result

$$\text{deg. } w_n(x) = \tfrac{1}{2}\phi(n)$$

TABLE 1. Factors and Proper Divisors of $A_{4n}(x)/x$ for $n = 2, 3, \ldots, 12$

n	$A_{4n}(x)/x$	$w_{4n}(x)$	Other factor(s)
2	$x^2 - 2$	$x^2 - 2$	$A_4(x)/x$
3	$x^4 - 4x^2 + 3$	$x^2 - 3$	$A_6(x)$
4	$x^6 - 6x^4 + 10x^2 - 4$	$x^4 - 4x^2 + 2$	$A_8(x)/x$
5	$x^8 - 8x^6 + 21x^4 - 20x^2 + 5$	$x^4 - 5x^2 + 5$	$A_{10}(x)$
6	$x^{10} - 10x^8 + 36x^6 - 56x^4 + 35x^2 - 6$	$x^4 - 4x^2 + 1$	$w_8(x) \cdot A_{12}(x)/x$
7	$x^{12} - 12x^{10} + 55x^8 - 70x^6 + 126x^4 - 56x^2 + 7$	$x^6 - 7x^4 + 14x^2 - 7$	$A_{14}(x)$
8	$x^{14} - 14x^{12} + 78x^{10} - 220x^8 + 330x^6 - 252x^4 + 84x^2 - 8$	$x^8 - 8x^6 + 20x^4 - 16x^2 + 2$	$A_{16}(x)/x$
9	$x^{16} - 16x^{14} + 105x^{12} - 364x^{10} + 715x^8 - 792x^6 + 462x^4 - 120x^2 + 9$	$x^6 - 6x^4 + 9x^2 - 3$	$w_{12}(x) \cdot A_{18}(x)$
10	$x^{18} - 18x^{16} + 136x^{14} - 560x^{12} + 1365x^{10} - 2002x^8 + 1716x^6 - 792x^4 + 165x^2 - 10$	$x^8 - 8x^6 + 19x^4 - 12x^2 + 1$	$w_8(x) \cdot A_{20}(x)/x$
11	$x^{20} - 20x^{18} + 171x^{16} - 816x^{14} + 2380x^{12} - 4368x^{10} + 5005x^8 - 3432x^6 + 1287x^4 - 220x^2 + 11$	$x^{10} - 11x^8 + 44x^6 - 77x^4 + 55x^2 - 11$	$A_{22}(x)$
12	$x^{22} - 22x^{20} + 210x^{18} - 1140x^{16} + 3876x^{14} - 8568x^{12} + 12376x^{10} - 11440x^8 + 6435x^6 - 2002x^4 + 286x^2 - 12$	$x^8 - 8x^6 + 20x^4 - 16x^2 + 1$	$w_{16}(x) \cdot A_{24}(x)/x$

TABLE 2. List of Factors for $n = 2, 3, \ldots, 14$ (9 decimal places) from Wilson [9]

2.	a	2.000000000
3.	a	3.000000000
	b	1.000000000
4.	a	3.414213562
	b	2.000000000
	c	0.585786437
5.	a	3.618033989
	b	2.618033989
	c	1.381966010
	d	0.381966010
6.	a	3.732050807
	b	3.000000000
	c	2.000000000
	d	1.000000000
	e	0.267949192
7.	a	3.801937736
	b	3.246979612
	c	2.445041864
	d	1.554958135
	e	0.753020387
	f	0.198062263
8.	a	3.847759064
	b	3.414213562
	c	2.765366862
	d	2.000000000
	e	1.234633137
	f	0.585786437
	g	0.152240935
9.	a	3.879385241
	b	3.532088884
	c	3.000000000
	d	2.347296348
	e	1.652703651
	f	1.000000000
	g	0.467911115
	h	0.120614758
10.	a	3.902113033
	b	3.618033989
	c	3.175570503
	d	2.618033989
	e	2.000000000
	f	1.381966010
	g	0.824429496
	h	0.381966010
	i	0.097886966
11.	a	3.918985948
	b	3.682507069
	c	3.309721461
	d	2.830830027
	e	2.284629680
	f	1.715370319
	g	1.169169972
	h	0.690278538
	i	0.317492930
	j	0.081014051
12.	a	3.931851652
	b	3.732050807
	c	3.414213562
	d	3.000000000
	e	2.517638088
	f	2.000000000
	g	1.482361911
	h	1.000000000
	i	0.585786437
	j	0.267949192
	k	0.068148347
13.	a	3.941883635
	b	3.770912051
	c	3.497021494
	d	3.136129492
	e	2.709209771
	f	2.241073362
	g	1.758926637
	h	1.290790228
	i	0.863870507
	j	0.502978505
	k	0.229087948
	l	0.058116364
14.	a	3.949855824
	b	3.801937736
	c	3.563662962
	d	3.246979612
	e	2.867767476
	f	2.445041864
	g	2.000000000
	h	1.554958135
	i	1.132232523
	j	0.753020387
	k	0.436337037
	l	0.198062263
	m	0.050144175

where $\phi(n)$ is Euler's ϕ-function, is useful. E.g., deg. $w_{20}(x) = 4 = \frac{1}{2}\phi(20)$.

From (1.2) and (2.3),

$$\frac{A_{4n}(x)}{x} = \prod_{d|4n} w_d(x) \qquad n \geq 2, \text{ since } w_4(x) = x$$

(2.6) $$= \prod_{j=1}^{n-1} (x^2 - \beta_j)$$

whence

$$\left.\frac{A_{4n}(x)}{x}\right|_{x=0} = \prod_{d|4n} w_d(0) \qquad n \geq 2$$

(2.7) $$= (-1)^{n-1}\prod_{j=1}^{n-1}\beta_j \qquad \text{from (2.6)}$$

$$= (-1)^{n-1}n \qquad \text{from (2.4).}$$

Consider, as an example, the case $n = 5$, i.e.,

$$5 = \left.\frac{A_{20}(x)}{x}\right|_{x=0} = \prod_{j=1}^{4}\beta_j,$$

from (2.7). Then the factors of 5 are given by the β_i of

$$x^4 - 5x^2 + 5 = (x^2 - \tfrac{1}{2}(5 + \sqrt{5}))(x^2 - \tfrac{1}{2}(5 - \sqrt{5})) = w_{20}(x)$$
$$= (x^2 - 3.618033989)(x^2 - 1.381966010)$$
$$= (x^2 - \beta_1)(x^2 - \beta_3)$$

and

$$(x^2 - x - 1)(x^2 + x - 1) = x^4 - 3x^2 + 1 = w_{10}(x)w_5(x)$$
$$= (x^2 - \tfrac{1}{2}(3 + \sqrt{5}))(x^2 - \tfrac{1}{2}(3 - \sqrt{5}))$$
$$= (x^2 - 2.618033989)(x^2 - 0.381966010)$$
$$= (x^2 - \beta_2)(x^2 - \beta_4),$$

that is,

$$5 = \beta_1\beta_2\beta_3\beta_4$$

where the subscript labelling of the irrational β's has been chosen to correspond to the decreasing order of magnitude given by Wilson [9], and where numerical calculations have been computed by pocket calculator to nine decimal places.

Our β_i have a simple trigonometrical expression. From [8] and (1.4),

(2.8) $$A_{2n}(2x) = U_{n-1}(x) \qquad n \geq 2,\ U_0 = 1$$

where $U_n(x)$ is the *Chebyshev polynomial of the second kind* (Magnus, Oberhettinger, and Soni [7]). That is,

(2.9) $$\frac{A_{4n}(2x)}{x} = \frac{U_{2n-1}(x)}{x} \qquad n \geq 1.$$

Solving

(2.10) $$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = 0$$

for θ gives

$$\theta = \frac{k\pi}{n+1} \qquad (k = 0, 1, 2, \ldots, n).$$

Therefore, the $n-1$ β_i of (2.2) are simply

$$\beta_i = 4\cos^2\frac{i\pi}{2n} \qquad (i = 1, 2, \ldots, n-1). \tag{2.11}$$

Of course, (2.4) with the β_i given by (2.11), is a known result (see, e.g., Durell and Robson [1]).

In the example following (2.7), where $n = 5$, we have

$$\beta_1 = 4\cos^2\frac{\pi}{10},\ \beta_2 = 4\cos^2\frac{\pi}{5},\ \beta_3 = 4\cos^2\frac{3\pi}{10},\ \beta_4 = 4\cos^2\frac{2\pi}{5}.$$

Wilson's $a, b, c, \ldots$ in Table 2 are $\beta_1, \beta_2, \beta_3, \ldots$.

Clearly, from (2.11),

$$\beta_i + \beta_{n-i} = 4. \tag{2.12}$$

Polynomials $A_{2n-1}(x)$ satisfy the identity previously established in [5], namely,

$$A_{2n+1}(x) = A_{2n+2}(x) + A_{2n}(x), \tag{2.13}$$

so the polynomials $A_n(x)$ for n odd are the sum of two consecutive Chebyshev polynomials.

Moreover,

$$A_{2n+1}(x) = \overline{f}_n(x) \tag{2.14}$$

in the notation of Hancock [3], about which further comments will be made later.

3. GENERATION OF IRRATIONAL FACTORS OF INTEGERS

One of our main results is Theorem 1 (below) relating to the system of equations satisfied by the β_i $(= \alpha_i^2)$.

Lemma 1:

$$n\delta(2, n) = \sum_{j=0}^{n-2}(-1)^j\binom{2n-j-1}{j}2^{2n-2j-3} \tag{3.1}$$

in which

$$\delta(2, n) = \begin{cases}1 & \text{if } 2\mid n,\\ 0 & \text{if } 2\nmid n.\end{cases} \tag{3.2}$$

Proof: Equation (1.72) of Gould [2] states that

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n-k}{k}2^{n-2k} = n + 1.$$

Algebraic manipulation of this equation yields

$$2n = \sum_{j=0}^{n-1}(-1)^j\binom{2n-j-1}{j}2^{2n-2j-1}$$

so

$$n = \sum_{j=0}^{n-1}(-1)^j\binom{2n-j-1}{j}2^{2n-2j-2}$$

$$= (-1)^{n-1}n + \sum_{j=0}^{n-2}(-1)^j\binom{2n-j-1}{j}2^{2n-2j-2},$$

that is,

$$n + (-1)^n n = \sum_{j=0}^{n-2}(-1)^j\binom{2n-j-1}{j}2^{2n-2j-2},$$

whence

$$n\,(2,\,n) = \sum_{j=0}^{n-2} (-1)^j \binom{2n-j-1}{j} 2$$

since $n + (-1)^n n = 2n\delta(2,\,n)$.

Theorem 1: The $n - 1$ α^2 of $\dfrac{A_{4n}(x)}{x} = 0$ satisfy the system of equations

$$(3.3)\qquad \begin{cases} \alpha_1^2 - \alpha_2^2 + \cdots + (-1)\,\alpha_{n-1}^2 = 2 \\ \alpha_1^4 - \alpha_2^4 + \cdots + (-1)\,\alpha_{n-1}^4 = 2^3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_1^{2n-2} - \alpha_2^{2n-2} + \cdots + (-1)^n \alpha_{n-1}^{2n-2} = 2^{2n-3}. \end{cases}$$

Proof: To solve (2.2), consider the $n - 1$ α_i^2 $(i = 1, 2, \ldots, n - 1)$. Then

$$0 = \sum_{i=1}^{n-1} (-1)^{i-1} A_{4n}(\alpha_i)/\alpha_i$$

$$= \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} (-1)^{i+j-1} \binom{2n-j-1}{j} \alpha_i^{2n-2j-2} \qquad \text{from (1.4)}$$

$$= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} (-1)^{i+j-1} \binom{2n-j-1}{j} \alpha_i^{2n-2(j+1)}$$

$$= \sum_{j=0}^{n-2} \sum_{i=1}^{n-1} (-1)^{i+j-1} \binom{2n-j-1}{j} \alpha_i^{2n-2(j+1)} + \sum_{i=1}^{n-1} (-1)^{i+n-2} \binom{n}{n-1} \alpha_i^0$$

$$= \sum_{j=0}^{n-2} (-1)^j \binom{2n-j-1}{j} \sum_{i=1}^{n-1} (-1)^{i-1} \alpha_i^{2n-2(j+1)} - n\delta(2,\,n) \qquad \text{by (3.2)}$$

$$= \sum_{j=0}^{n-2} (-1)^j \binom{2n-j-1}{j} \sum_{i=1}^{n-1} (-1)^{i-1} \alpha_i^{2n-2(j+1)} - \sum_{j=0}^{n-2} (-1)^j \binom{2n-j-1}{j} 2^{2n-2j-3}$$

by Lemma 1

from which it follows that, with a slight variation in the set of values of j,

$$\sum_{i=1}^{n-1} (-1)^{i-1} \alpha_i^{2n-2j} = 2^{2n-2j-1} \qquad j = 1, 2, \ldots, n - 1,$$

which is the system of equations (3.3).

Illustration of Theorem 1: Theorem 1 tells us that there are 2 β_i of

$$\frac{A_{12}(x)}{x} = x^4 - 4x^2 + 3 = 0$$

which satisfy (3.3) when $n = 3$, i.e., $\beta_1 - \beta_2 = 2$, $\beta_1^2 - \beta_2^2 = 2^3$, namely,

$$(3.4)\qquad \beta_1 = 3 = 4\cos^2\frac{\pi}{6},\ \beta_2 = 1 = 4\cos^2\frac{\pi}{3}.$$

Also, there are 5 β_i of $\dfrac{A_{24}(x)}{x} = 0$ which satisfy (3.3) when $n = 6$, i.e.,

$$\sum_{i=1}^{5} (-1)^{i-1} \beta_i^j = 2^{2j-1} \qquad j = 1, 2, \ldots, 5,$$

namely,

$$(3.5)\quad \begin{cases} \beta_1 = 3.732050807 = 2 + \sqrt{3} = 4\cos^2 \frac{\pi}{12},\ \beta_2 = 3.000000000, \\ \beta_3 = 2.000000000,\ \beta_4 = 1.000000000,\ \beta_5 = 0.267949192 = 2 - \sqrt{3}, \end{cases}$$

as can be seen in the entry for $n = 6$ in Table 2.

Similarly, there are 8 β_i of $\frac{A_{36}(x)}{x} = 0$ which satisfy (3.3) when $n = 9$, i.e.,

$$\sum_{i=1}^{8} (-1)^{i-1}\beta_i^j = 2^{2j-1} \qquad j = 1, 2, \ldots, 8,$$

namely,

$$(3.6)\quad \begin{cases} \beta_1 = 3.8793385241 = 4\cos^2 \frac{\pi}{18},\ \beta_2 = 3.532088884,\ \beta_3 = 3.000000000, \\ \beta_4 = 2.347296348,\ \beta_5 = 1.652703651,\ \beta_6 = 1.000000000, \\ \beta_7 = 0.467911115,\ \beta_8 = 0.120614758, \end{cases}$$

as can be seen in the entry for $n = 9$ in Table 2.

From (3.4), (3.5), and (3.6), we observe that

$$\begin{aligned} 3 &= \beta_1\beta_2 && n = 3 \\ &= \beta_1\beta_2\beta_4\beta_5 && n = 6 \\ &= \beta_1\beta_2\beta_4\beta_5\beta_7\beta_8 && n = 9 \end{aligned}$$

(and so on). Notice that every β_i, for which $3|i$, does not occur in the products. This is the gist of (3.9). (Other combinations are possible, e.g.,

$$\begin{aligned} 3 &= \beta_2\beta_4 && n = 6 \\ &= \beta_3\beta_6 && n = 9 \\ &= \beta_4\beta_8 && n = 12 \end{aligned}$$

and so on.)

Elementary trigonometry with (2.6) and (2.11) may be used to show that

$$(3.7)\quad A_{4n}(x)/x + (-1)^n\left\{A_{4n}((4 - x^2)^{\frac{1}{2}})\right\}/x = 0$$

where, by the second term in (3.7) is meant the expression for $A_{4n}(x)/x$ when x^2 is replaced by $4 - x^2$.

If (3.7) is treated from a combinatorial number theory point of view, we have, on using (1.4) and the binomial expansion for $(4 - x^2)^{n-1-j}$ and then considering the coefficient of $x^{2n-2-2p}$, the result

$$(3.8)\quad \sum_{j=0}^{n-1} (-1)^j 2^{2(p-j)} \binom{2n-1-j}{j}\binom{n-1-j}{p-j} = \binom{2n-1-p}{p}$$

for every $p \le n - 1$.

This identity is very similar to result (3.44) in Gould [2].

The next (known) result is important for Table 2:

$$(3.9)\quad \prod_{\substack{i=1 \\ r\nmid i}}^{n-1} \beta_i = r \qquad r|n,\ \beta_{j+1} < \beta_j\ (j = 1, \ldots, n - 2).$$

To prove (3.9), divide (2.4) by $\prod_{i=1}^{k-1} \beta_{ri} = \prod_{i=1}^{k-1} \beta_i^* = k$ where $\beta_i^* = 4\cos^2 \frac{i\pi}{2k}$ and

$n = rk$, i.e., $r|n$, i.e., $r|2n$. E.g., $n = 8$ in Table 2 gives

$$2 = \prod_{\substack{i=1 \\ 2 \nmid i}}^{7} \beta_i = \beta_1\beta_3\beta_5\beta_7 \qquad \text{with } 2 = \beta_2\beta_6 = \beta_4.$$

Refer also to the Illustration of Theorem 1 on page 244 above.

From (3.9), and, earlier, (3.4), (3.5), and (3.6), it is clear that the sequence (1.1) shows how any integer (>0) may be expressed as a product of (mostly) irrational numbers in an infinite number of ways, in accordance with a pattern of generation.

4. MISCELLANEOUS RESULTS

Results (4.1)-(4.5), which are stated without proof, may be derived from (1.2) and (2.11).

$$\text{(4.1)} \qquad \frac{A_{4n}(x)}{x} = \begin{cases} A_{2n}(x) \cdot B_{4n}(x) & n \text{ odd} \\ \dfrac{A_{2n}(x)}{x} \cdot B_{4n}(x) & n \text{ even} \end{cases}$$

where $B_{4n}(x) = w_{4n}(x) \times$ (some product of proper divisors depending on the factors of n).

Some particular instances of (4.1) are shown in Table 1.

Consider again the transformation $x^2 \to 4 - x^2$. This has the following effects:

n odd

(4.2) $\beta_{2i} \to \beta_{2i-1}$ in reverse order (and conversely), so

$$\text{(4.3)} \qquad A_{2n}(x) \leftrightarrow B_{4n}(x)$$

n even

$$\text{(4.4)} \qquad \frac{A_{2n}(x)}{x} \leftrightarrow \frac{A_{2n}(x)}{x}$$

$$\text{(4.5)} \qquad B_{4n}(x) \leftrightarrow B_{4n}(x).$$

Previously, in (2.14), we mentioned the connection between our $A_{2n+1}(x)$ and $\overline{f}_n(x)$ in Hancock [3]. It is instructive to compare in detail our treatment, where the motivation originated from combinatorial and number theoretic considerations, with Hancock's approach to somewhat similar material through cyclotomy and trigonometry.

However, to conserve space, we merely indicate without justification some comparisons of interest as well as some fresh properties of $A_n(x)$. Familiarity with Hancock's notation is assumed.

Observe, firstly that our

$$A_{2n}(x),\ A_{4n+2}(x),\ B_{4(2n+1)}(x), \text{ and } xB_{4(2n+1)}(x) + 2$$

are, respectively, Hancock's

$$A_{n-1}(x),\ \psi_{2n}(x),\ \Phi_{2n}(x), \text{ and } F_{2n+1}(x).$$

Further, we note that

$$(4.6)\qquad \begin{cases} A_{2n+2}(x) - A_{2n}(x) = f_n(x) = (-1)^n \overline{f}_n(-x) \\ A_{2n}(x) = \tfrac{1}{2}(f_{n-1}(x) + \overline{f}_{n-1}(x)) \\ \sum_{k=1}^{n} f_k(x) = A_{2n+2}(x) - 1 \end{cases}$$

while some fresh results are

$$(4.7)\qquad \begin{cases} A_{4n+2}(x) = A_{2n+2}^2(x) - A_{2n}^2(x) \\ A_{2n}(2) = n \\ A_{2n+1}(2) = n + 1. \end{cases}$$

5. CONCLUDING COMMENTS

Newton's iteration can be used to solve the system of equations (3.3). Alternatively, the problem may be approached through the theory of recurring sequences.

Using the notation of Jarden [6], we may consider equation (2.2), with x replaced by $\sqrt{y}$, as the auxiliary equation of the homogeneous linear recurrence relation of order $n - 1$:

$$(5.1)\qquad 0 = \sum_{j=0}^{n-1} (-1)^j \binom{2n - j - 1}{j} w_{m-j}^{(n-1)},$$

where

$$(5.2)\qquad w_m^{(n-1)} = \sum_{i=1}^{n-1} (-1)^{i-1} \beta_i^m$$

is the general term of the recurring sequence $\{w_m^{(n-1)}\}$ defined by (5.1) with the initial conditions (3.3). Thus, when $n = 3$, (2.2) becomes

$$x^4 - 4x^2 + 3 = 0$$

which can be rewritten as

$$y^2 - 4y + 3 = 0$$

i.e., the auxiliary equation for (5.1) in the form

$$w_m^{(2)} = 4w_{m-1}^{(2)} - 3w_{m-2}^{(2)}.$$

Initial conditions are

$$w_1^{(2)} = \beta_1 - \beta_2 = 2$$

and

$$w_2^{(2)} = \beta_1^2 - \beta_2^2 = 2^3.$$

Finally, it is worth noting that the theoretical foundations for the ideas implicit in [9] and [10] have by no means been fully exploited.

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A HISTORY OF THE FIBONACCI Q-MATRIX AND A HIGHER-DIMENSIONAL PROBLEM

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To the memory of Verner E. Hoggatt, Jr.

One of the most popular and recurrent recent methods for the study of the Fibonacci sequence is to define the so-called Fibonacci Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

so that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{2}$$

where $F_{n+1} = F_n + F_{n-1}$, with $F_1 = 1$, $F_0 = 0$.

Theorems may then be cited from linear algebra so as to give speedy proofs of Fibonacci formulas. Write $|A|$ for the determinant of a matrix A. Then it is well known that $|AB| = |A| \cdot |B|$, and in general $|A^n| = |A|^n$. The Fibonacci Q-matrix method then gives at once the famous formula

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{3}$$

which was first given by Robert Simson in 1753. Formula (3) is the basis for the well-known geometrical paradox attributed to Lewis Carroll in which a unit of area mysteriously appears or disappears upon dissecting a suitable square and reassembling into a rectangle.

Where did this Q-matrix method originate? The object of the present paper is to give a tentative answer to this question, and present a reasonably complete bibliography of papers bearing on the use of such a matrix for the study of Fibonacci numbers. An unsolved problem is included.

The phrase "Q-matrix" seems to have originated in the master's thesis of Charles King [10]. At least, Basin and Hoggatt [16] cite this source, and from then on the idea caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in our *Fibonacci Quarterly* authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became a central tool in the analysis of Fibonacci properties. Vern Hoggatt carried on a far-ranging correspondence in which he jotted down ideas and made innumerable suggestions for further research. For example, his letters to me make up a foot-high stack of paper very nearly, representing creative thinking going on for 20