IDENTITIES FOR CERTAIN PARTITION FUNCTIONS AND THEIR DIFFERENCES*

ROBERT D. GIRSE
Idaho State University, Pocatello, ID 83209

1. INTRODUCTION

If $i \geq 0$ and $n \geq 1$, let $q_i^e(n)$ $(q_i^o(n))$ denote the number of partitions of into an even (odd) number of parts, where each part occurs at most i times; $q_i^e(0) = 1$ $(q_i^o(0) = 0)$. If $i \geq 0$ and $n \geq 0$, let $\Delta_i(n) = q_i^e(n) - q_i^o(n)$.

We note that for $i \geq 0$ and $n \geq 0$, $q_i^e(n) + q_i^o(n) = p_i(n)$, where $p_i(n)$ denotes the number of partitions of n where each part occurs at most i times.

The purpose of this paper is to give identities for $q_i^e(n)$, $q_i^o(n)$, and $\Delta_i(n)$. The function $\Delta_i(n)$ has been studied by Hickerson [3] and [4], and by Alder and Muwafi [1]. They have given formulas to determine $\Delta_i(n)$, for $i > 1$, in terms of certain restricted partition functions. The case $i = 1$ is a well known result due to Euler [2, p. 285]. Another result of this type, the Sylvester-Euler theorem [5, p. 264], states

$$\Delta(n) = (-1)^n Q(n), \tag{1}$$

where $\Delta(n)$ is the difference function with the restriction on the number of times a part may occur removed, and $Q(n)$ is the number of partitions of n into distinct odd parts.

Here we first obtain identities for $\Delta_i(n)$, some of which are recursive. We then find several identities for $q_i^e(n)$ and $q_i^o(n)$ which also give us some new results for $\Delta_i(n)$. Our identities not only demonstrate relationships between these functions and other partition functions, but many of them are also useful computationally.

We will make use of the following partition functions in addition to those already defined. For $n \geq 1$:

(i) $p(n)$ $(q(n))$ denotes the number of (distinct) partitions of n.

(ii) $p_{a_1, \ldots, a_r; b}(n)$ $(q_{a_1, \ldots, a_r; b}(n))$ denotes the number of (distinct) partitions of n into parts $\equiv a_j \pmod{b}$, $1 \leq j \leq r$.

(iii) $Q_k(n)$ denotes the number of partitions of n into distinct odd multiples of k.

(iv) $q^i(n)$; $p^i_{0;2}(n)$ denote, respectively, the number of partitions of n into distinct parts and even parts, where no part is divisible by i.

By convention, when $n = 0$, each of these partition functions assumes the value 1.

We let $[x]$ denote the greatest integer function and $\sum_r$ denote the sum over all nonnegative r such that the summands are defined. Finally, we let m be an integer ≥ 1 unless otherwise specified.

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2. IDENTITIES FOR THE DIFFERENCE FUNCTION

We will base our proofs in this section on the generating function of Δ_i, which is given by

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n = \prod_{j=1}^{\infty} \frac{1 + (-1)^i x^{(i+1)j}}{1 + x^j} \tag{2}$$

Theorem 1: (i) $\Delta_{2m}(n) = \sum_{r=0}^{n} \Delta(r) q_{0;\,2m+1}(n - r)$,

(ii) $\Delta_{2m-1}(n) = \sum_{r} (-1)^r \Delta(n - (3r^2 \pm r)m)$.

Proof: Since

$$\sum_{n=0}^{\infty} \Delta(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1 + x^j} \tag{3}$$

and

$$\sum_{n=0}^{\infty} q_{0;\,a}(n)x^n = \prod_{j=1}^{\infty} (1 + x^{aj}), \tag{4}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_{2m}(n)x^n &= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1}{1 + x^j} \prod_{j=1}^{\infty} (1 + x^{(2m+1)j}) \\ &= \sum_{n=0}^{\infty} \Delta(n)x^n \sum_{n=0}^{\infty} q_{0;\,2m+1}(n)x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \Delta(r) q_{0;\,2m+1}(n - r) \right) x^n,\end{aligned}$$

and equating coefficients proves (i). On the other hand,

$$\sum_{n=0}^{\infty} \Delta_{2m-1}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1}{1 + x^j} \prod_{j=1}^{\infty} (1 - x^{2mj}).$$

Now Euler's Pentagonal Number Theorem [2, p. 284] states

$$\prod_{j=1}^{\infty} (1 - x^{aj}) = \sum_{r=-\infty}^{\infty} (-1)^r x^{\frac{1}{2}(3r^2 + r)a}. \tag{5}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_{2m-1}(n)x^n &= \sum_{n=0}^{\infty} \Delta(n)x^n \sum_{r=0}^{\infty} (-1)^r x^{(3r^2 \pm r)m} \\ &= \sum_{n=0}^{\infty} \left(\sum_{r} (-1)^r \Delta(n - (3r^2 \pm r)m) \right) x^n,\end{aligned}$$

and equating coefficients gives (ii).

Using the Sylvester-Euler identity (1) for Δ in Theorem 1 yields the following result.

Corollary 1: (i) $\Delta_{2m}(n) = \sum_{r=0}^{n} (-1)^r Q(r) q_{0;\,2m+1}(n - r)$,

(ii) $\Delta_{2m-1}(n) = (-1)^n \sum_{r} (-1)^r Q(n - (3r^2 \pm r)m)$.

Theorem 2: (i) $\Delta_{4m-1}(n) = \sum_{r=0}^{n} \Delta_{2m-1}(r) q_{0;\,2m}(n - r)$,

(ii) $\Delta_{4m+1}(n) = \sum_{r} (-1)^r \Delta_{2m}(n - \frac{1}{2}(3r^2 \pm r)(2m + 1))$,

where (ii) also holds for $m = 0$.

Proof: From (2) we have

$$\sum_{n=0}^{\infty} \Delta_{4m-1}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{4mj}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 + x^j} \prod_{j=1}^{\infty}(1 + x^{2mj}).$$

Thus, applying (2) and (4),

$$\sum_{n=0}^{\infty} \Delta_{4m-1}(n)x^n = \sum_{n=0}^{\infty} \Delta_{2m-1}(n)x^n \sum_{n=0}^{\infty} q_{0;\,2m}(n)x^n,$$

and (i) follows. Now

$$\sum_{n=0}^{\infty} \Delta_{4m+1}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(4m+2)j}}{1 + x^j} = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 + x^j} \prod_{j=1}^{\infty}(1 - x^{(2m+1)j})$$

$$= \sum_{n=0}^{\infty} \Delta_{2m}(n)x^n \sum_{r=0}^{n} (-1)^r x^{\frac{1}{2}(3r^2 \pm r)(2m+1)}$$

from (2) and (5), and (ii) follows immediately.

Theorem 3:

$$\sum_{r=0}^{n} \Delta_i(n)q(n - r) = \begin{cases} q_{0;\,2m+1}(n) & \text{for } i = 2m, \\ \left.\begin{cases} (-1)^r & \text{if } n = (3r^2 \pm r)m \text{ for } r = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}\right\} & \text{for } i = 2m - 1. \end{cases}$$

Proof: Using (2) and (4) we have

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \sum_{n=0}^{\infty} q(n)x^n = \prod_{j=1}^{\infty} (1 + (-1)^i x^{(i+1)j}).$$

Thus,

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_i(r)q(n - r)\right)x^n = \begin{cases} \prod_{j=1}^{\infty}(1 + x^{(2m+1)j}) = \sum_{n=0}^{\infty} q_{0;\,2m+1}(n)x^n & \text{for } i = 2m, \\ \prod_{j=1}^{\infty}(1 - x^{2mj}) = \sum_{r=0}^{\infty} (-1)^r x^{(3r^2 \pm r)m} & \text{for } i = 2m - 1, \end{cases}$$

From (4) and (5). Equating coefficients, the theorem is proved.

Theorem 4:

$$\sum_{r=0}^{n} \Delta_i(r)p_i(n - r) = \begin{cases} p_{0;\,2}^{4m+2}(n) & \text{for } i = 2m, \\ \sum_{r} (-1)^r p_{0;\,2}^{2m}(n - (3r^2 \pm r)m) & \text{for } i = 2m - 1. \end{cases}$$

Proof: The generating function of p_i is given by

$$\sum_{n=0}^{\infty} p_i(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(i+1)j}}{1 - x^j}, \tag{6}$$

and so using this and (2):

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \sum_{n=0}^{\infty} p_i(n)x^n = \prod_{j=1}^{\infty} \frac{1 + (-1)^i x^{(i+1)j}}{1 - x^{2j}}(1 - x^{(i+1)j}).$$

Now if $i = 2m$,

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_{2m}(r)p_{2m}(n - r)\right)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)}}{1 - x^{2j}}(1 - x^{(2m+1)j})$$

$$= \prod_{j=1}^{\infty} \frac{1 - x^{2(2m+1)j}}{1 - x^{2j}}$$

$$= \prod_{\substack{j \geq 1 \\ 2m+1 \nmid j}} \frac{1}{1 - x^{2j}} = \sum_{n=0}^{\infty} p_{0;\,2}^{4m+2}(n)x^n.$$

Likewise, if $i = 2m - 1$,

$$\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \Delta_{2m-1}(r)p_{2m-1}(n - r)\right)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^{2j}}(1 - x^{2mj})$$

$$= \sum_{n=0}^{\infty} p_{0;\,2}^{2m}(n)x^n \sum_{r=0}^{\infty} (-1)^r x^{(3r^2 \pm r)m}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{r} (-1)^r p_{0;\,2}^{2m}(n - (3r^2 \pm r)m)\right)x^n,$$

where we use (5) to obtain the second equation. Thus the theorem is proved.

Corollary 2: If n is odd,

$$\sum_{r=0}^{n} \Delta_i(r)p_i(n - r) = 0.$$

Proof: First we note that both $p_{0;\,2}^{4m+2}(n) = 0$ and $p_{0;\,2}^{2m}(n) = 0$ for $n \equiv 1 \pmod 2$, and since $n - (3r^2 \pm r)m \equiv n \pmod 2$ the corollary follows from Theorem 4.

Theorem 5: For $n \geq 1$,

(i) $\displaystyle\sum_{r=0}^{n} \Delta_{2m}(r)q^{2m+1}(n - r) = 0,$

(ii) $\displaystyle\sum_{r=0}^{n} \Delta_{2m-1}(r)p_{0,1,3,\ldots,2m-1;\,2m}(n - r) = 0.$

Proof: From (2) we have

$$\sum_{n=0}^{\infty} \Delta_i(n)x^n \prod_{j=1}^{\infty} \frac{1 + x^j}{1 + (-1)^i x^{(i+1)j}} = 1,$$

where

$$\prod_{j=1}^{\infty} \frac{1 + x^j}{1 + x^{(2m+1)j}} = \prod_{\substack{j \geq 1 \\ 2m+1 \nmid j}} (1 + x^j) = \sum_{n=0}^{\infty} q^{2m+1}(n)x^n,$$

and

$$\prod_{j=1}^{\infty} \frac{1 + x^j}{1 - x^{2mj}} = \prod_{j=1}^{\infty} \frac{1}{(1 - x^{2mj})(1 - x^{2j-1})}$$

$$= \prod_{j=0}^{\infty} \frac{1}{(1 - x^{2mj+1})(1 - x^{2mj+3}) \cdots (1 - x^{2mj+(2m-1)})(1 - x^{2mj+2m})}$$

$$= \sum_{n=0}^{\infty} p_{0,1,3,\ldots,2m-1;\,2m}(n)x^n,$$

and so the theorem follows.

3. IDENTITIES FOR THE DEFINING PARTITION FUNCTIONS

We will base the proofs in this section on the generating functions of q_i^e and q_i^o, which we construct in the following two lemmas.

Lemma 1: (i) $$\sum_{n=0}^{\infty} q_{2m}^e(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty}(-1)^{r+1}x^{(2m+1)(r+1)^2} + \sum_{r=0}^{\infty}(-1)^r x^{r^2}\right),$$

(ii) $$\sum_{n=0}^{\infty} q_{2m}^o(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty}(-1)^r x^{(2m+1)r^2} - \sum_{r=0}^{\infty}(-1)^r x^{r^2}\right).$$

Proof: First we recall that $p_i(n) = q_i^e(n) + q_i^o(n)$. Thus, using the definition of $\Delta_i(n)$, we have $2q_i^e(n) = p_i(n) + \Delta_i(n)$. Hence

$$2\sum_{n=0}^{\infty} q_{2m}^e(n)x^n = \sum_{n=0}^{\infty} p_{2m}(n)x^n + \sum_{n=0}^{\infty} \Delta_{2m}(n)x^n,$$

and so, from (2) and (6), we have

$$2\sum_{n=0}^{\infty} q_{2m}^e(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 - x^j} + \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 + x^j}$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 + x^{(2m+1)j}} + \prod_{j=1}^{\infty} \frac{1 - x^j}{1 + x^j}\right).$$

Now

$$\prod_{j=1}^{\infty} \frac{1 - x^{aj}}{1 + x^{aj}} = \sum_{r=-\infty}^{\infty} (-1)^r x^{ar^2},$$

which is a special case of Jacobi's identity [2, p. 283]. Using this result twice yields,

$$2\sum_{n=0}^{\infty} q_{2m}^e(n)x^n = \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=-\infty}^{\infty} (-1)^r x^{(2m+1)r^2} + \sum_{r=-\infty}^{\infty} (-1)^r x^{r^2}\right)$$

$$= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(2 + 2\sum_{r=1}^{\infty}(-1)^r x^{(2m+1)r^2} + 2\sum_{r=1}^{\infty}(-1)^r x^{r^2}\right),$$

from which (i) follows immediately. To prove (ii), we note that

$$q_{2m}^{o}(n) = p_{2m}(n) - q_{2m}^{e}(n),$$

and so

$$\begin{aligned}\sum_{n=0}^{\infty} q_{2m}^{o}(n)x^n &= \sum_{n=0}^{\infty} p_{2m}(n)x^n - \sum_{n=0}^{\infty} q_{2m}^{e}(n)x^n \\ &= \prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 - x^j} - \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^{r+1} x^{(2m+1)(r+1)^2} \right. \\ &\qquad \left. + \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right) \\ &= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\prod_{j=1}^{\infty} \frac{1 - x^{(2m+1)j}}{1 + x^{(2m+1)j}} - \sum_{r=0}^{\infty} (-1)^{r+1} x^{(2m+1)(r+1)^2} \right. \\ &\qquad \left. - \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right) \\ &= \prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j}\left(\sum_{r=-\infty}^{\infty} (-1)^r x^{(2m+1)r^2} - \sum_{r=1}^{\infty} (-1)^r x^{(2m+1)r^2} \right. \\ &\qquad \left. - \sum_{r=0}^{\infty} (-1)^r x^{r^2}\right).\end{aligned}$$

Simplifying the right-hand side of this equation yields (ii), and so the lemma is proved.

Using the same method of proof as in Lemma 1, with several minor alterations, proves the following result.

Lemma 2: (i) $\displaystyle\sum_{n=0}^{\infty} q_{2m-1}^{e}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^r x^{r^2}\right)$,

(ii) $\displaystyle\sum_{n=0}^{\infty} q_{2m-1}^{o}(n)x^n = \prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j}\left(\sum_{r=0}^{\infty} (-1)^r x^{(r+1)^2}\right)$.

We now give identities for q_i^e and q_i^o, and then combine these results to obtain formulas for Δ_i. First note that in Lemma 1, using (4),

$$\prod_{j=1}^{\infty} \frac{1 + x^{(2m+1)j}}{1 - x^j} = \sum_{n=0}^{\infty} q_{0;\,2m+1}(n)x^n \sum_{n=0}^{\infty} p(n)x^n,$$

and in Lemma 2, from (6),

$$\prod_{j=1}^{\infty} \frac{1 - x^{2mj}}{1 - x^j} = \sum_{n=0}^{\infty} p_{2m-1}(n)x^n.$$

Thus, using these two results, the following two theorems follow directly from the lemmas.

Theorem 6:

(i) $$q_{2m}^{e}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^{r+1}\big(p(k - (2m+1)(r+1)^2) \\ \qquad\qquad - p(k - r^2)\big), \\ \displaystyle\sum_{k=0}^{n} p(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^{r+1}\big(q_{0;\,2m+1}(k - (2m+1)(r+1)^2) \\ \qquad\qquad - q_{0;\,2m+1}(k - r^2)\big), \end{cases}$$

$$\text{(ii)}\quad q^{\mathrm{o}}_{2m}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^r (p(k-(2m+1)r^2) - p(k-r^2)), \\ \displaystyle\sum_{k=0}^{n} p(n-k) \sum_{r=0}^{[\sqrt{k}]} (-1)^r (q_{0;\,2m+1}(k-(2m+1)r^2) \\ \qquad\qquad - q_{0;\,2m+1}(k-r^2)), \end{cases}$$

where $p(n) = 0$ and $q_{0;\,2m+1}(n) = 0$ for $n < 0$.

Theorem 7: (i) $q^{\mathrm{e}}_{2m-1}(n) = \displaystyle\sum_{r=0}^{[\sqrt{n}]} (-1)^r p_{2m-1}(n-r^2)$,

(ii) $q^{\mathrm{o}}_{2m-1}(n) = \displaystyle\sum_{r=1}^{[\sqrt{n}]} (-1)^{r+1} p_{2m-1}(n-r^2)$.

Using the results of Theorems 6 and 7 and the definition of Δ_i proves the following corollary.

Corollary 3:

$$\text{(i)}\quad \Delta_{2m}(n) = \begin{cases} \displaystyle\sum_{k=0}^{n} q_{0;\,2m+1}(n-k)\left(p(k) + 2\sum_{r=1}^{[\sqrt{k}]} (-1)^r p(k-r^2)\right), \\ \displaystyle\sum_{k=0}^{n} p(n-k)\left(q_{0;\,2m+1}(k) + 2\sum_{r=1}^{[\sqrt{k}]} (-1)^r q_{0;\,2m+1}(k-r^2)\right), \end{cases}$$

$$\text{(ii)}\quad \Delta_{2m-1}(n) = p_{2m-1}(n) + 2\sum_{r=1}^{[\sqrt{n}]} (-1)^r p_{2m-1}(n-r^2).$$

Corollary 4: $\Delta(n) = p(n) + 2\displaystyle\sum_{r=1}^{[\sqrt{n}]} (-1)^r p(n-r^2)$.

Proof: This follows from the results of Theorem 1(i) and Corollary 3(i).

Multiplying both sides of the generating functions in Lemma 1 by

$$\prod_{j=1}^{\infty} (1 - x^j),$$

and using (5), yields the following identities.

Theorem 8:

$$\text{(i)}\quad \sum_k (-1)^k q^{\mathrm{e}}_{2m}(n - \tfrac{1}{2}(3k^2 \pm k)) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r+1} (q_{0;\,2m+1}(n - (2m+1)(r+1)^2) - q_{0;\,2m+1}(n-r^2)),$$

$$\text{(ii)}\quad \sum_k (-1)^k q^{\mathrm{o}}_{2m}(n - \tfrac{1}{2}(3k^2 \pm k)) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r} (q_{0;\,2m+1}(n - (2m+1)r^2) - q_{0;\,2m+1}(n-r^2)),$$

where $q_{0;\,2m+1}(n) = 0$ when $n < 0$.

Corollary 5:

$$\sum_k (-1)^k \Delta_{2m}(n - \tfrac{1}{2}(3k^2 \pm k)) = q_{0;\,2m+1}(n) + 2\sum_{r=1}^{[\sqrt{n}]} (-1)^r q_{0;\,2m+1}(n-r^2).$$

Theorem 9:

$$\text{(i)}\quad \sum_{r=0}^{n} (-1)^{n-r} q^{\mathrm{e}}_{2m}(r) Q_{2m+1}(n-r) = \sum_{r=0}^{[\sqrt{n}]} (-1)^{r+1} (p(n - (2m+1)(r+1)^2) - p(n-r^2)),$$

$$\text{(ii)}\quad \sum_{r=0}^{n}(-1)^{n-r}q_{2m}^{o}(r)Q_{2m+1}(n-r) = \sum_{r=0}^{[\sqrt{n}]}(-1)^{r}(p(n-(2m+1)r^2)-p(n-r^2)),$$

where $p(n) = 0$ when $n < 0$.

Proof: This follows from Lemma 1 if we multiply the generating functions on both sides by

$$\prod_{j=1}^{\infty}\frac{1}{1+x^{(2m+1)j}} = \prod_{j=1}^{\infty}(1-x^{(2m+1)(2j-1)}) = \sum_{n=0}^{\infty}(-1)^{n}Q_{2m+1}(n)x^{n}.$$

Theorem 10: (i) $\displaystyle\sum_{r=0}^{n}q_{2m-1}^{e}(r)p_{0;\,2m}(n-r) = \sum_{r=0}^{[\sqrt{n}]}(-1)^{r}p(n-r^2),$

(ii) $\displaystyle\sum_{r=0}^{n}q_{2m-1}^{o}(r)p_{0;\,2m}(n-r) = \sum_{r=1}^{[\sqrt{n}]}(-1)^{r+1}p(n-r^2).$

Proof: Here we multiply the generating functions of Lemma 2 by

$$\prod_{j=1}^{\infty}\frac{1}{1-x^{2mj}} = \sum_{n=0}^{\infty}p_{0;\,2}(n)x^{n} \tag{7}$$

on both sides, and the theorem follows.

Corollary 6:

$$\left.\begin{array}{l}\displaystyle\sum_{r=0}^{n}(-1)^{n-r}\Delta_{2m}(r)Q_{2m+1}(n-r)\\ \displaystyle\sum_{r=0}^{n}\Delta_{2m-1}(r)p_{0;\,2m}(n-r)\end{array}\right\} = \Delta(n).$$

Proof: Using the results of Theorems 9 and 10, we have

$$\left.\begin{array}{l}\displaystyle\sum_{r=0}^{n}(-1)^{n-r}\Delta_{2m}(r)Q_{2m+1}(n-r)\\ \displaystyle\sum_{r=0}^{n}\Delta_{2m-1}(r)p_{0;\,2m}(n-r)\end{array}\right\} = p(n) + 2\sum_{r=1}^{[\sqrt{n}]}(-1)^{r}p(n-r^2),$$

and so this result follows from Corollary 4.

Theorem 11:

$$\text{(i)}\quad \sum_{k=0}^{n}p_{0;\,2m}(n-k)\sum_{r}(-1)^{r}q_{2m-1}^{e}(k-\tfrac{1}{2}(3r^2\pm r)) = \begin{cases}(-1)^{n} \text{ if } n=t^2,\\ \quad \text{for } t=0,1,2,\ldots,\\ 0 \text{ otherwise,}\end{cases}$$

$$\text{(ii)}\quad \sum_{k=0}^{n}p_{0;\,2m}(n-k)\sum_{r}(-1)^{r}q_{2m-1}^{o}(k-\tfrac{1}{2}(3r^2\pm r)) = \begin{cases}(-1)^{n} \text{ if } n=t^2,\\ \quad \text{for } t=1,2,\ldots,\\ 0 \text{ otherwise.}\end{cases}$$

Proof: These identities follow from Lemma 2 if we multiply both sides of the generating functions by

$$\prod_{j=1}^{\infty}\frac{1-x^{j}}{1-x^{2mj}},$$

and use (5) and (7).

Corollary 7:

$$\sum_{k=0}^{n} p_{0;\,2m}(n-k)\sum_{r}(-1)^r\Delta_{2m-1}(k-\tfrac{1}{2}(3r^2 \pm r)) = \begin{cases} 1 \text{ if } n = 0 \\ (-1)^n 2 \text{ if } n = t^2, \\ \quad \text{for } t = 1, 2, \ldots, \\ 0 \text{ otherwise.} \end{cases}$$

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FIBONACCI AND LUCAS NUMBERS OF THE FORMS $w^2 - 1$, $w^3 \pm 1$

NEVILLE ROBBINS

Bernard M. Baruch College, New York, NY 10010

INTRODUCTION

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. All such numbers of the forms w^2, w^3, $w^2 + 1$ have been determined by J. H. E. Cohn [2], H. London and R. Finkelstein [8], R. Finkelstein [4] and [5], J. C. Lagarias and D. P. Weisser [7], R. Steiner [10], and H. C. Williams [11]. In this article, we find all Fibonacci and Lucas nubmers of the forms $w^2 - 1$, $w^3 \pm 1$.

PRELIMINARIES

(1) $L_n = w^2 \rightarrow n = 1$ or 3

(2) $L_n = 2w^2 \rightarrow n = 0$ or ± 6

(3) $L_n = w^3 \rightarrow n = \pm 1$

(4) $L_n = 2w^3 \rightarrow n = 0$

(5) $L_n = 4w^3 \rightarrow n = \pm 3$

(6) $L_{-n} = (-1)^n L_n$

(7) $(F_n, F_{n-1}) = (L_n, L_{n-1}) = 1$

(8) $3|F_n$ iff $4|n$

(9) $L_{2n} = L_n^2 - 2(-1)^n$

(10) $L_{2n+1} = L_n L_{n+1} - (-1)^n$

(11) If $(x, y) = 1$ and $xy = w^n$, then $x = u^n$, $y = v^n$, with $(u, v) = 1$ and $uv = w$.

(12) $F_{4n\pm 1} = F_{2n\pm 1}L_{2n} - 1$

(13) $F_{4n} = F_{2n-1}L_{2n+1} - 1$

(14) $F_{4n-2} = F_{2n-2}L_{2n} - 1$