ON SQUARE LUCAS NUMBERS

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Among the first dozen members of the Lucas sequence (1, 3, 4, 7, 11, 18, ...) there are two squares, $L_1 = 1$ and $L_3 = 4$. Are there any other squares in the Lucas sequence?

Since the period of the Lucas sequence modulo 8 is 12, it follows that $L_{12k+\lambda} \equiv L_\lambda \pmod 8$, so that all possible residues are represented in the following table.

λ	$L_{12k+\lambda} \pmod 8$
0	2
1	1
2	3
3	4
4	7
5	3
6	2
7	5
8	7
9	4
10	3
11	7

It follows that the only Lucas numbers which may be squares are $L_{12k+\lambda}$ with $\lambda = 1, 3$ or 9, since the other residues modulo 8 are quadratic non-residues of 8.

From the general relation

$$2\,L_{a+b} = 5\,F_a F_b + L_a L_b$$

it follows if $t = 2^r$, $r \ge 1$, that

$$\begin{aligned} 2\,L_{\lambda+2t} &= 5\,F_\lambda F_{2t} + L_\lambda L_{2t} \\ &= 5\,F_\lambda F_t L_t + L_\lambda (L_t^2 - 2) \end{aligned}$$

so that

$$2\,L_{\lambda+2t} \equiv -2\,L_\lambda \pmod{L_t}$$

But $(L_t, 2) = 1$. Hence

$$L_{\lambda+2t} \equiv -L_\lambda \pmod{L_t}$$

We can use this relation to advantage by writing

$$L_{12k+\lambda} \text{ as } L_{\lambda+2mt}$$

where m is odd and $t = 2^r$, $r \geq 1$.

Then

$$\begin{aligned} L_{\lambda+2mt} &\equiv -L_{\lambda+2(m-1)t} \pmod{L_t} \\ &\equiv +L_{\lambda+2(m-2)t} \pmod{L_t} \\ &\quad \cdots\cdots\cdots\cdots \\ &\equiv (-1)^m L_\lambda \pmod{L_t} \end{aligned}$$

For $\lambda = 1$,

$$L_{12k+1} \equiv -L_1 \equiv -1 \pmod{L_t}, \quad t = 2^r, \quad r \geq 1$$

But

$$\left(\frac{-1}{L_t}\right)^* = -1, \quad \text{since } L_t \equiv 3 \pmod 4$$

Therefore L_{12k+1} may not be a perfect square except for $L_1 = 1$. Similarly, L_{12k+3} can be shown to be ruled out by entirely the same argument except for $L_3 = 4$.

Finally,

$$L_{12k+9} = L_{4k+3}\,[L^2_{4k+3} + \theta\,]$$

The θ in the bracket may be either 3 or 1. But since only Lucas numbers L_{4k+2} are divisible by 3, it follows that L_{4k+3} and $L^2_{4k+3} + 3$ are relatively prime. Therefore, if L_{12k+9} is to be a perfect square, both factors must be such. It is clear that L_{4k+3} is not a perfect square for $k = 1$ or 2. For other values, k equals either $3k'$, $3k' + 1$ or $3k' + 2$ with $k' \geq 1$. But this gives us Lucas numbers $L_{12k'+3}$, $L_{12k'+7}$, and $L_{12k'+11}$ respectively and it has already been shown that these cannot be squares.

Thus the only squares in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$.

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*Legendre's symbol.