ON THE GENERAL TERM OF A RECURSIVE SEQUENCE

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INTRODUCTION

It is often comforting and useful to obtain a specific formula for the general term of a recursive sequence. This paper reviews the Fibonacci and Lucas sequences, then presents a more general method which requires the solution of a set of linear equations. The solution may be effected by finding the inverse of a Vandermonde matrix, and a description of this inverse is included.

THE SPECIAL CASE

As is well known, the Fibonacci sequence is completely defined by the difference equation $F(n) - F(n-1) - F(n-2) = 0$ and the initial conditions $F(0) = 0$ and $F(1) = 1$. If we seek a solution of the difference equation of the form $F(n) = x^n$, we obtain $x^n - x^{n-1} - x^{n-2} = 0$, or $x^2 - x - 1 = 0$. This has two solutions; $x_1 = (1 + \sqrt{5})/2$ and $x_2 = (1 - \sqrt{5})/2$. Now, the theory of homogeneous linear difference equations assures us that the most general solution is $F(n) = c_1x_1^n + c_2x_2^n$, where c_1 and c_2 are arbitrary constants. (The reader who encounters this result for the first time can verify it by substitution; the theory parallels quite nicely the theory of linear differential equations.)

The initial conditions give us two linear equations,

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1x_1 + c_2x_2 &= 1 \end{aligned} \tag{1}$$

The solutions are $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, and we obtain the well-known formula $F(n) = 1/\sqrt{5}(x_1^n - x_2^n)$.

The Lucas series is obtained from the same difference equation with different initial conditions. In this case, $F(0) = 2$, $F(1) = 1$, and equations (1) become

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1x_1 + c_2x_2 &= 1 \quad . \end{aligned}$$

Then the general term of the Lucas sequence is

$$L(n) = \left[\frac{1+\sqrt{5}}{2}\right]^n + \left[\frac{1-\sqrt{5}}{2}\right]^n .$$

From these formulas it is possible to prove such identities as $L(n) - F(n) = 2F(n-1)$ and $L(n) + F(n) = 2F(n+1)$.

THE GENERAL CASE

We might solve all equations of the form of equations (1) by writing them in matrix form

$$(2) \qquad \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} F(0) \\ F(1) \end{bmatrix}$$

and then find the inverse of the first matrix, so that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} F(0) \\ F(1) \end{bmatrix} .$$

The matrix

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}$$

is a simple case of a Vandermonde matrix, and the determination of the constants is possible if the matrix can be inverted. Fortunately it can be easily inverted, even if the order exceeds two.

Let us suppose that a recursion relation gives birth to a linear homogenous difference equation with constant coefficients, say,

$F(n) + a_1 F(n-1) + \cdots a_k F(n-k) = 0$, and that $F(0) = b_0$, $F(1) = b_1, \cdots$, $F(k-1) = b_{k-1}$.

If we seek again a solution of the form $F(n) = x^n$, then we are led to the equation

$$f(x) = x^k + a_1 x^{k-1} + \cdots + a_k = 0 \; . \tag{3}$$

Then assuming that the roots of (3), $x_1, x_2, \cdots, x_k$ are all different*, the theory of difference equations assures us that $F(n) = c_1 x_1^n + c_2 x_2^n + \cdots + c_k x_k^n$. The subsequent equations corresponding to (1), but written in matrix form are

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_1^{k-1} & x_2^{k-1} & & x_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_k \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{k-1} \end{bmatrix} \tag{4}$$

or $V_k C = B$.

Now the polynomial $f_1(x) = f(x)/(x - x_1)$ has k coefficients, and moreover $f_1(x_2) = f_1(x_3) = \cdots = f_1(x_k) = 0$. Consequently if we form a row vector, (written in reverse order) of these coefficients, then this row vector will be orthogonal to every column of V_k except the first. We need now only a normalizing factor, so that the scalar product of this row vector with the first column of V_k is unity. Investigation shows that this scalar product (before normalizing) is $f_1(x_1) = f'(x_1)$, the first derivative of $f(x)$ at $x = x_1$. Moreover, the fact that $f'(x_1) = \lim_{x \to x_1} \frac{f(x)}{x - x_1}$ makes this scalar product easy to calculate by synthetic division.

This procedure is now continued; the coefficients of $f_2(x)/f'(x_2)$ provide us with the second row of V_k^{-1}, and in general the coefficients of $f_i(x)/f'(x_i)$ provide the i^{th} row of V_k^{-1}.

A particular example makes the procedure clear. Suppose the recurrence relation is

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*When there are multiple roots, the matrix takes a different form; the inverse for this case is not presented here.

$$F(n) - 3F(n-1) - 5F(n-2) + 15F(n-3) + 4F(n-4) - 12F(n-5) = 0,$$

and the initial conditions are

$$F(0) = 1,\ F(1) = 1,\ F(2) = 1,\ F(3) = 2$$
$$F(4) = 3.$$

The difference equation yields the polynomial

$$x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0,$$

whose roots are $[-2, -1, 1, 2, 3]$ The coefficients of $f_1(x)$ are easily found by synthetic division, the normalizing factor by repeated synthetic division.

$$\begin{array}{r|llllll}
-2 & 1 & -3 & -5 & +15 & +4 & -12 \\
 & & -2 & +10 & -10 & -10 & +12 \\
\hline
-2 & 1 & -5 & +5 & +5 & -6 & \quad 0 \\
 & & -2 & +14 & -38 & +66 & \\
\hline
 & 1 & -7 & +19 & -33 & +60 &
\end{array}$$

The vector $(-6, 5, 5, -5, 1)$ is orthogonal to all the columns of V_5 except the first, the normalizing factor is $1/60$, and the first row of V_5^{-1} is

$$\left(\frac{-1}{10}, \frac{1}{12}, \frac{1}{12}, \frac{-1}{12}, \frac{1}{60}\right)$$

Synthetic division may be continued for the other roots until we obtain the desired inverse.

$$V_5^{-1} = \begin{bmatrix}
-\frac{1}{10} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{60} \\
\frac{1}{2} & -\frac{2}{3} & \frac{1}{24} & \frac{1}{6} & -\frac{1}{24} \\
1 & \frac{2}{3} & -\frac{7}{12} & \frac{1}{6} & \frac{1}{12} \\
-\frac{1}{2} & -\frac{1}{12} & \frac{7}{12} & \frac{1}{12} & -\frac{1}{12} \\
\frac{1}{10} & 0 & -\frac{1}{8} & 0 & \frac{1}{40}
\end{bmatrix}$$

Multiplying equation (4) on the left by V_5^{-1}, we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{60} \\ \frac{1}{2} & -\frac{2}{3} & \frac{1}{24} & \frac{1}{6} & -\frac{1}{24} \\ 1 & \frac{2}{3} & \frac{7}{12} & \frac{1}{6} & \frac{1}{12} \\ -\frac{1}{2} & -\frac{1}{12} & \frac{7}{12} & \frac{1}{12} & -\frac{1}{12} \\ \frac{1}{10} & 0 & -\frac{1}{8} & 0 & \frac{1}{40} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{20} \\ \frac{1}{12} \\ 1 \\ \frac{1}{12} \\ \frac{1}{20} \end{bmatrix}$$

Hence the general term is given by

$$F(n) = -\frac{1}{20}(-2)^n + \frac{1}{12}(-1)^n + 1(1)^n - \frac{1}{12}(2)^n + \frac{1}{20}(3)^n .$$

CORRECTIONS FOR VOLUME 1, NO. 2

Page 4: Equation (2.8) should read

$$(a-b)^p \sum_{k=o}^{p} (-1)^k \binom{p}{k} \sum_{j=o}^{q} \binom{q}{j} F(a^{p+q-k-j}\, b^{k+j} x) = \sum_{n=0}^{\infty} A_n x^n F_n^p L_n^p$$

Page 23: The fifth line up from the bottom should read:

$$D_0 = 0,\ D_1 = x + y,\ D_2 = (x+y)^2 .$$

Page 30: In Line 10, replace $m(u_{n+1} - 1)$ by $m \mid (u_{n+1} - 1)$.

Page 33: The = signs in lines 10 and 11 should be replaced by $\equiv$ signs.

Page 37: The first line of the title should end in a lower case "m."