Then

$$S_n = b^{(r-1)n} \sum_{N_o=0}^{r-1} \sum_{N_1=0}^{r-1} \cdots \sum_{N_{n-1}=0}^{r-1} \left(\frac{a}{b}\right)^{N_o+N_1+\ldots N_{n-1}}$$

$$= \prod_{m=0}^{n-1} \left\{ b^{r-1} \sum_{N_m=0}^{r-1} \left(\frac{a}{b}\right)^{N_m} \right\} = \prod_{m=1}^{n} \left\{ \frac{a^r - b^r}{a-b} \right\} = \left(\frac{a^r-b^r}{a-b}\right)^n$$

If

$$a = \frac{1}{2}(1 + \sqrt{5}), \; b = \frac{1}{2}(1 - \sqrt{5}), \text{ then } \frac{a^r - b^r}{a-b} = F_r ,$$

the Fibonacci number. In that case,

$$S_n\left(r, \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 - \sqrt{5})\right) = F_r^n .$$

Also solved by the proposer.

XXXXXXXXXXXXXXX

A DIGIT MUSES*.........

Oh!
4
2B
No zero
In the world of math!
Would that I were like that great
Built into the structure of the universe and art
The ideal of ideals dividing all things in proportions of gold — a paragon!

Brother U. Alfred

*This poem has the distinction that the number of syllables in each line proceeds by the sequence: 1, 1, 2, 3, 5, 8, 13, 21.

XXXXXXXXXXXXXX