ZECKENDORF'S THEOREM AND SOME APPLICATIONS

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1. INTRODUCTION

The subject theorem, due to E. Zeckendorf[1], is one which deserves to be more widely known, particularly since the property involved in the theorem statement is a property which uniquely characterizes the Fibonacci numbers among all other sequences of positive integers. Our purpose in this paper is to give a brief exposition of theorem with its proof, and to examine several applications and consequences.

For the subsequent proof, it is convenient to define the Fibonacci numbers $\{u_n\}_1^\infty$ as follows: $u_1 = 1$, $u_2 = 2$, $u_{n+1} = u_n + u_{n-1}$ for $n \geq 2$. If we take $\{F_n\}_1^\infty$ according to the more common definition, $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, then $u_n = F_{n+1}$ for $n \geq 1$.

Zeckendorf's theorem essentially states that every positive integer can be represented uniquely as a finite sum of distinct Fibonacci numbers $\{u_n\}$, with the additional constraint that no two consecutive Fibonacci numbers appear in the representation of any particular integer. A formal statement of the theorem and its proof follow in section 2, while section 3 is concerned with applications and a converse.

2. ZECKENDORF'S THEOREM

Theorem: Every positive integer N has one and only one representation in the form

(1) $$N = \sum_{1}^{\infty} \alpha_i u_i$$

where each α_i is a binary digit and

(2) $$\alpha_i \alpha_{i+1} = 0 \quad \text{for} \quad i \geq 1.$$

(In the following, we shall reserve the subscripted variables α and β for binary digits, that is, digits which have either the value zero or unit.)

The proof is accomplished with the aid of two lemmas:

Lemma 1: $$u_n = 1 + u_{n-1} + u_{n-3} + \ldots + u_{1,2},$$

where
$$u_{1,2} = \begin{cases} u_1 & \text{if } n \text{ is odd} \\ u_2 & \text{if } n \text{ is even.} \end{cases}$$

Proof: The elementary inductive verification of this identity is left to the reader.

Lemma 2: Representation of a positive integer in the form (1) with binary coefficients satisfying (2) is unique.

Proof: Assume $\exists$ a positive integer N with two distinct representations of the required form, so that

$$N = \sum_1^{\infty} \alpha_i u_i = \sum_1^{\infty} \beta_i u_i \tag{3}$$

with $\alpha_i \alpha_{i+1} = \beta_i \beta_{i+1} = 0$ for $i \geq 1$, and

$$\sum_1^{\infty} |\alpha_i - \beta_i| \neq 0.$$

Let k be the largest integer i such that $\alpha_i \neq \beta_i$; then of the two quantities α_k and β_k, one must be unity and the other zero. Assume without loss of generality that $\alpha_k = 1$, $\beta_k = 0$, so that (3) becomes

$$\sum_1^{k} \alpha_i u_i = \sum_1^{k-1} \beta_i u_i . \tag{4}$$

But the left-hand side is $\geq u_k$ since $\alpha_k = 1$, while the right-hand side satisfies

$$\sum_1^{k-1} \beta_i u_i \leq u_{k-1} + u_{k-3} + \ldots + u_{1,2} = u_k - 1 ,$$

a contradiction. We conclude $\alpha_i = \beta_i$ for all $i \geq 1$; that is, the representation is unique.

Proof of Theorem:

It remains to be shown that every positive integer N has a representation in the form (1) with binary coefficients satisfying (2).

We will prove, by an induction on n, that $0 \leq N < u_n$ implies

$$N = \sum_{1}^{n-1} \alpha_i u_i \quad \text{with} \quad \alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1.$$

The statement is vacuuously true for $n = 1$ and is verified by inspection for $n = 2$ and $n = 3$. Now, assume the proposition has been proved for $n = 1, 2, \ldots, k$ where k is some integer ≥ 3; we wish to show the statement must necessarily be true for $n = k+1$, or equivalently, that $0 \leq N < u_{k+1}$ implies

$$N = \sum_{1}^{k} \alpha_i u_i \quad (\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1). \tag{5}$$

By the induction hypothesis, the result holds for N in the range $0 \leq N < u_k$, so that we need only consider the case $u_k \leq N < u_{k+1}$. For this latter case,

$$0 \leq N - u_k < u_{k+1} - u_k = u_{k-1},$$

and the induction hypothesis guarantees binary coefficient β_i such that

$$N - u_k = \sum_{1}^{k-2} \beta_i u_i . \quad (\beta_i \beta_{i+1} = 0 \quad \text{for } i \geq 1) .$$

Transposing the u_k, we obtain

$$N = \sum_{1}^{k-2} \beta_i u_i + u_k ,$$

so that the choices, $\alpha_i = \beta_i$ for $1 \leq i \leq k-2$, $\alpha_{k-1} = 0$ and $\alpha_k = 1$, yield a representation in the required form (5). q. e. d.

3. APPLICATIONS AND A RELATED PROBLEM

As our first application, we analyse the problem[2] of determining the probability that at least two successive "heads" will occur in n flips of a fair coin. To investigate this problem, we consider the complementary situation and ask for the number of ways in which a coin can be tossed n times without ever getting two heads in sequence. Clearly, this number is equal to the number of distinct binary sequences $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of length n, where each α_i is either 1 (heads) or 0 (tails), with the additional constraint that a 1 is never followed immediately by another 1. This latter condition is concisely expressed by the requirement $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$, which, of course, is exactly the coefficient condition of the preceding section.

Let us term a sequence of n binary digits an "admissible" sequence if it satisfies the constraint $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$; then, we wish to determine, as a function of n, the number of admissible sequences.

To each admissible sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, let us associate the number

$$\sum_{i=1}^{n} \alpha_i u_i$$

so that a one-to-one correspondence,

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \longleftrightarrow \sum_{1}^{n} \alpha_i u_i ,$$

is established between admissible sequences (of length n) and a subset of the positive integers. But, from the proof of Zeckendorf's theorem, we have that each integer N satisfying $0 \leq N < u_{n+1}$ has one and only one representation in the form

$$\sum_{1}^{n} \alpha_i u_i$$

with $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$, and clearly no integer $\geq u_{n+1}$ can be represented in this form. Hence, the number of different integers which can be represented is equal to the number of integers in the set $\{0, 1, 2, \ldots, u_{n+1} - 1\}$, or u_{n+1}. By our correspondence, the number of admissible sequences of length n is therefore also u_{n+1}. Since the total number of binary sequences of length n is 2^n, the probability of not obtaining at least two successive heads in n throws is

$$\frac{u_{n+1}}{2^n},$$

or, equivalently, the required probability of having at least two successive heads in n tosses is

$$1 - \frac{u_{n+1}}{2^n}.$$

A second application may be found in Whinihan's recent paper[3] on determining an optimum strategy for the game of Fibonacci Nim. In developing the strategy, the author introduces a rule for representing an arbitrary integer as a unique sum of distinct Fibonacci numbers, so that in the sequence of expansion coefficients, it is "impossible for two 1's to appear...without at least one 0 separating them." As noted in an editorial comment, this unique representation property is precisely the content of the Zeckendorf theorem.

Lastly, we consider the unique representation property in Zeckendorf's theorem and ask what other integer sequences (if any), in addition to the Fibonacci sequence, enjoy the same property. For clarity, we define the property in question as follows:

Definition: A sequence of positive integers $\{v_n\}_1^\infty$ is said to possess the <u>unique representation property</u> (u. r. p.) if and only if every positive integer N has a <u>unique</u> representation in the form

$$N = \sum_1^\infty \alpha_i u_i, \tag{6}$$

where the α_i are binary digits satisfying

$$\alpha_i \alpha_{i+1} = 0 \quad \text{for } i \geq 1. \tag{7}$$

The main theorem concerning u. r. p. sequences is due to D. E. Daykin[4]:

Theorem (Daykin): If $\{v_n\}$ is a sequence possessing the u. r. p., then v_n is necessarily increasing and $v_n = u_n$ for all $n \geq 1$.

Thus, the Fibonacci sequence is the only sequence, increasing or otherwise, for which unique representations in the form (6)-(7) are possible for every positive integer.

Daykin's theorem is easy to prove in the case of increasing v_n but is non-trivial for the general case in which the v_n's may appear in any order. The general result provides a complete converse to Zeckendorf's theorem and also gives a concise characterization of the Fibonacci sequence as being the only sequence possessing the unique representation property.

A different, though related, characterization of the Fibonacci numbers in terms of "complete" sequences has been given earlier by the author[5]. Any sequence possessing the u. r. p. is, a fortiori, complete; that is, every positive integer may be written as a sum of distinct members of the sequence. Moreover, it can be shown that the deletion of any single term from a u. r. p. sequence renders the remaining sequence incomplete. The definition of completeness, unlike that of the u. r. p., is invariant with respect to a reordering of the sequence and may provide an alternate method of proving Daykin's theorem. The connection between completeness and the unique representation property will be the subject of a future paper.

REFERENCES

1. C. G. Lekkerkerker, "Voorstelling van natuurlÿke getallen door een som van Fibonacci," Simon Stevin, 29 (1951-52), 190-195.

2. Problem 62-6, SIAM Review, 4 (1962), 255.

3. M. J. Whinihan, "Fibonacci Nim," The Fibonacci Quarterly, 1 (1963)3, 9-13.

4. D. E. Daykin, "Representation of natural numbers as sums of generalized Fibonacci numbers," Journal of the London Mathematical Society, 35 (1960), 143-160.

5. J. L. Brown, Jr., "Note on Complete Sequences of Integers," American Mathematical Monthly, 68 (1961), 557-560.

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