ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-44 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that for every positive integer k there are no more than n Fibonacci numbers between n^k and n^{k+1}.

B-45 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let H_n be the n-th generalized Fibonacci number, i.e., let H_1 and H_2 be arbitrary and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Show that $nH_1 + (n-1)H_2 + (n-2)H_3 + \ldots + H_n = H_{n+4} - (n+2)H_2 - H_1$.

B-46 *Proposed by C.A. Church, Jr., Duke University, Durham, North Carolina*

Evaluate the n-th order determinant

$$D_n = \begin{vmatrix} a+b & ab & 0 & 0 & \ldots \\ 1 & a+b & ab & 0 & \ldots \\ 0 & 1 & a+b & ab & \ldots \\ 0 & 0 & 1 & a+b & \ldots \\ \ldots & & & & \\ \ldots & & & & \\ \ldots & & & & \end{vmatrix} .$$

B-47 *Proposed by Barry Litvack, University of Michigan, Ann Arbor, Michigan*

Prove that for every positive integer k there are k consecutive Fibonacci numbers each of which is composite.

B-48 *Proposed by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada*

Prove that

$$\sum_{k=1}^{r-1} (-2)^k \binom{r}{k} F_k = \begin{cases} -2^r F_r & \text{if } r \text{ is an even positive integer} \\ 2^r F_r - 2(5)^{(r-1)/2} & \text{if } r \text{ is an odd positive integer,} \end{cases}$$

where $F_{n+2} = F_{n+1} + F_n (F_1 = F_2 = 1)$ and find the corresponding sum in which the F_k are replaced by the Lucas numbers L_k.

B-49 *Proposed by Anton Glaser, Pennsylvania State University, Abington, Pennsylvania*

Let ϕ represent the <u>letter</u> "oh".	TWϕ
Given that T, W, ϕ, L, V, P, and TWϕ are	IS
Fivonacci numbers, solve the cryptarithm	THE
in the base 14, introducing the digits	ϕNLY
α, β, ∂, and δ in base 14 for 10, 11,	EVEN
12, and 13 in base 10.	PRIME

B-50 *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that

$$\sum_{j=0}^{n} \left[2F_j^2 - \binom{n}{j} F_j \right] = F_n^2 .$$

B-51 *Proposed by Douglas Lind, Falls Church, Virginia*

Let $\phi(n)$ be the Euler totient and let $\phi^k(n)$ be defined by $\phi^1(n) = \phi(n)$, $\phi^{k+1}(n) = \phi\left[\phi^k(n)\right]$. Prove that $\phi^n(F_n) = 1$, where F_n is the n-th Fibonacci number.

SOLUTIONS

A PERIODIC RECURRENT SEQUENCE

B-30 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California*

Find the millionth term of the sequence a_n given that

$$a_1 = 1, \; a_2 = 1, \text{ and } a_{n+2} = a_{n+1} - a_n \text{ for } n \geq 1 \quad .$$

Solution by J.A.H. Hunter, Toronto, Ontario, Canada

It is simple to show that a_n has a period of 6, with:

$$a_{6k+4} = a_{6k+5} = -1 \quad .$$

$10^6 \equiv 4 (\text{mod } 6)$, hence the millionth term must be -1.

Also solved by Charles R. Wall, Texas Christian University, Ft. Worth, Texas; John H. Halton, University of Colcrado, Boulder, Colorado; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; Vassili Daiev, Sea Cliff, L.I., N.Y.; George Ledin, Jr., San Francisco, California; Ronald Weinshenk, San Jose State College, San Jose, California; Dermott A. Breault, Sylvania—A.R.L., Waltham, Mass.; David E. Zitarelli, Temple University, Philadelphia, Pennyslvania; B. Litvack, University of Michigan, Ann Arbor, Michigan; and the proposer.

SUMS OF CONSECUTIVE FIBONACCI NUMBERS

B-31 *Proposed by Douglas Lind, Falls Church, Virginia*

If n is even, show that the sum of $2n$ consecutive Fibonacci numbers is divisible by F_n.

Solution by Roseanna Torretto, University of Santa Clara, Santa Clara, California

Let T be the sum $F_{a+1} + \ldots + F_{a+2n}$ of $2n$ consecutive Fibonacci numbers. Let $S_n = F_1 + F_2 + \ldots + F_n$. It is well known that $S_n = F_{n+2} - 1$. Hence

$$T = S_{a+2n} - S_n = F_{a+2n+2} - F_{a+2} \quad .$$

Since $F_{q+p} - F_{q-p} = L_q F_p$ for p even (see I. D. Ruggles, Some Fibonacci Results using Fibonacci-Type Sequences, this Quarterly, Vol. 1, No. 2, p. 77), $T = L_{a+n+2} F_n$ as desired.

Also solved by J.L. Borwn, Jr., Pennsylvania State University, State College, Pennsylvania; B. Litvack, University of Michigan, Ann Arbor, Michigan; John H. Halton, University of Colorado, Boulder, Colorado; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer.

A CONGRUENCE RELATION

B-32 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $nL_n \equiv F_n \pmod 5$.

Solution by John Allen Fuchs, University of Santa Clara, California

It follows from basic results on homogeneous linear difference equations that the sequence $Y_n = nL_n - F_n$ satisfies

$$Y_{n+4} = 2Y_{n+3} + Y_{n+2} - 2Y_{n+1} - Y_n \,, \tag{1}$$

i.e., $(E^2 - E - 1)^2 Y = 0$ with the operator E defined as in James A. Jeske, Linear Recurrence Relations — Part I, this Quarterly, Vol. 1, No. 2. The desired result now follows by trial for $n = 1, 2, 3$, and 4 and mathematical induction using (1).

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; Douglas Lind, Falls Church, Virginia; and the proposer.

TERM BY TERM SUMS

B-33 *Proposed by John A. Fuchs, University of Santa Clara, Santa Clara, California*

Let $u_n, v_n, \ldots, w_n$ be sequences each satisfying the second order recurrence formula

$$y_{n+2} = gy_{n+1} + hy_n \; (n \geq 1) \,,$$

where g and h are constants. Let $a, b, \ldots, c$ be constants. Show that

$$au_n + bv_n + \ldots + cw_n = 0$$

is true for all positive integral values of n if it is true for $n = 1$ and $n = 2$.

Solution by B. Litvack, University of Michigan, Ann Arbor, Michigan; John H. Halton, University of Colorado, Boulder, Colorado; and the proposer.

Suppose that

$$(1) \qquad au_n + bv_n + \ldots + cw_n = 0$$

for $n = 1$ and $n = 2$. Multiplying the first case by h and the second by g, we see that

$$ahu_1 + bhv_1 + \ldots + chw_1 = 0 ,$$

$$agu_2 + bgv_2 + \ldots + cgw_2 = 0 .$$

Adding, we obtain

$$au_3 + bv_3 + \ldots + cw_3 = 0$$

since u_n, v_n, ... , w_n all satisfy

$$y_{n+2} = gy_{n+1} + hy_n .$$

Repeating the process (or, more formally, using mathematical induction) we verify that (1) holds for all n if it holds for $n = 1, 2$.

Also solved

JARDEN PRODUCTS

Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, California

Let u_n and v_n be any two sequences satisfying the second-order recurrence formula

$$(1) \qquad y_{n+2} = gy_{n+1} + hy_n$$

where g and h are constants. Show that the sequence of products $w_n = u_nv_n$ satisfies a third-order recurrence formula

$$(2) \qquad y_{n+3} = ay_{n+2} + by_{n+1} + cy_n$$

and find a, b, and c as functions of g and h.

Solution by the proposer.

Let r and s be the roots of the auxiliary polynomial $x^2 - gx - h$ of (1). We assume $r \neq s$; the case $r = s$ has the same result. Now $u_n = c_{11} r^n + c_{12} s^n$, $v_n = c_{21} r^n + c_{22} s^n$, and so $w_n = c_1 (r^2)^n + c_2 (rs)^n + c_3 (s^2)^n$. Hence the auxiliary polynomial of (2) is

$$x^3 - ax^2 - bx - c = (x - r^2)(x - rs)(x - s^2) = \left[x^2 - (r^2 + s^2)x + (rs)^2\right] (x - rs) =$$

$$\left[x^2 - (g^2 + 2h)x + h^2\right] (x + h) = x^3 - (g^2 + h)x^2 - (g^2 + h)hx + h^3 .$$

Now $a = g^2 + h$, $b = (g^2 + h)h$, and $c = -h^3$.

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; and Charles R. Wall, Texas Christian University, Ft. Worth, Texas. This problem is a special case of formulas of D. Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem (Israel), 1958, p. 43.

This problem is a special case of formulas of D. Jarden, Recurring Sequences, Riveon Lematematika, Jerusalem (Israel), 1958, p. 43.

AN ALTERNATING BINOMIAL TRANSFORM

B-35 *Proposed by J.L. Brown, Jr., Pennsylvania State University, University Park, Pennsylvania*

Prove that

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = 0$$

for r an odd positive integer and generalize.

Solution by H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada

We have the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} ,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2} .$$

Thus

$$1 - \alpha = \beta \quad \text{or} \quad 1 - \beta = \alpha \; .$$

Now

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = -\binom{r}{1} F_1 + \binom{r}{2} F_2 - \binom{r}{3} F_3 + \ldots + (-1)^{r-1} \binom{r}{r-1} F_{r-1}$$

$$= -\binom{r}{1}\left(\frac{\alpha - \beta}{\sqrt{5}}\right) + \binom{r}{2}\left(\frac{\alpha^2 - \beta^2}{\sqrt{5}}\right) - \binom{r}{3}\left(\frac{\alpha^3 - \beta^3}{\sqrt{5}}\right)$$

$$+ \ldots + (-1)^{r-1} \binom{r}{r-1}\left(\frac{\alpha^{r-1} - \beta^{r-1}}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{5}} \; -\binom{r}{1}\alpha + \binom{r}{2}\alpha^2 - \binom{r}{3}\alpha^3 + \ldots + (-1)^{r-1}\binom{r}{r-1}\alpha^{r-1}$$

$$+ \binom{r}{1}\beta - \binom{r}{2}\beta^2 + \binom{r}{3}\beta^3 + \ldots + (-1)^{r-1}\binom{r}{r-1}\beta^{r-1} \; .$$

Let r be an odd positive integer. Then

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = \frac{1}{\sqrt{5}}\left\{[(1 - \alpha)^r - 1 + \alpha^r] + [1 - (1 - \beta)^r - \beta^r]\right\}$$

$$= \frac{1}{\sqrt{5}}[\beta^r - 1 + \alpha^r + 1 - \alpha^r - \beta^r]$$

$$= 0 \; .$$

Let r be an even positive integer. Then

$$\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = \frac{1}{\sqrt{5}}\left\{[(1 - \alpha)^r - 1 - \alpha^r] + [1 - (1 - \beta)^r + \beta^r]\right\}$$

$$= \frac{1}{\sqrt{5}}[\beta^r - 1 - \alpha^r + 1 - \alpha^r + \beta^r]$$

$$= -2\left(\frac{\alpha^r - \beta^r}{\sqrt{5}}\right)$$

$$= -2F_r \; .$$

For Lucas numbers it can be shown by analogous methods that

$$\sum_{k=1}^{r-1} (-1)^{k+1} \binom{r}{k} L_k = \begin{cases} 2 & \text{if } r \text{ is even} \\ 2 - 2L_r & \text{if } r \text{ is odd.} \end{cases}$$

Also solved by John H. Halton, University of Colorado, Boulder, Colorado; Douglas Lind, Falls Church, Virginia; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer.

THE PELL SEQUENCE

B-36 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

The sequence 1, 2, 5, 12, 29, 70, ... is defined by $c_1 = 1$, $c_2 = 2$, and $c_{n+2} = 2c_{n+1} + c_n$ for all $n \geq 1$. Prove that c_{5m} is an integral multiple of 29 for all positive integers m.

Solution by Douglas Lind, Falls Church, Virginia

Since $c_5 = 29$, the solution follows at once from the more general fact that for the above defined sequence,

(1) $$c_m \mid c_{nm} .$$

We shall prove this more general assertion following N. N. Vorobyov (The Fibonacci Numbers, Heath, 1963).

We need first establish that

(2) $$c_{n+k} = c_{n-1}c_k + c_n c_{k+1} .$$

Proof is by induction on k. The cases $k = 1$, $k = 2$ are easily shown true. We then assume (2) true for k and $k + 1$. Hence

(3) $$c_{n+k} = c_{n-1}c_k + c_n c_{k+1} ,$$

(4) $$c_{n+k+1} = c_{n-1}c_{k+1} + c_n c_{k-2} .$$

Multiplying (4) by two and adding to (3), we obtain

$$c_{n+k+2} = c_{n-1}c_{k+2} + c_n c_{k+3} ,$$

completing the induction step and proving (2).

We now prove the general assertion (1) by induction using (2). (1) is obviously true for $n = 1$. Now assume c_{nm} is divisible by c_m, $n \geq 1$, and consider $c_{(n+1)m}$. By (2),

$$c_{(n+1)m} = c_{nm-1}c_m + c_{nm}c_{m+1} \quad .$$

The first term on the right is divisible by c_m, and by the induction hypothesis so is the last term. Applying the fundamental theorem of arithmetic, so also must be $c_{(n+1)m}$. This completes the induction step and the proof of (1).

Also solved by B. Litvack, University of Michigan, Ann Arbor, Michigan; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; John H. Halton, University of Colorado, Boulder, Colorado; Dermott A. Breault, Sylvania A.R.L., Waltham, Mass.; J.A.H. Hunter, Toronto, Ontario, Canada; H.H. Ferns, University of Victoria, Victoria, British Columbia, Canada; J.L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania; and the proposer.

HARMONIC DIVISION

B-37 *Proposed by Brother U. Alfred, St. Mary's College, California*

Given a line with a point of origin O and four positive positions A, B, C, and D with respect to O. If the line segments OA, OB, OC, and OD correspond respectively to four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2}, F_{n+3}, determine for which set(s) of Fibonacci numbers the points A, B, C, and D are in simple harmonic ratio, i.e.,

$$\frac{AB}{BC}\,\frac{AD}{DC} = -1 \quad .$$

Solution by John H. Halton, University of Colorado, Boulder, Colorado

O, A, B, C, D are five consecutive points on a line, with $OA=F_n$, $OB=F_{n+1}$, $OC=F_{n+2}$, $OD=F_{n+3}$. Thus $AB=F_{n+1}-F_n=F_{n-1}$, $BC=F_{n+2}-F_{n+1}=F_n$, $AD=F_{n+3}-F_n = (F_{n+2}+F_{n+1}) - (F_{n+2}-F_{n+1}) = 2F_{n+1}$, $DC=F_{n-2}-F_{n+3} = -F_{n+1}$. Thus $AD/DC = -2$, and $AB/BC = F_{n-1}/F_n$. If B and D divide A and C harmonically, $(AB/BC)(AD/DC) = -1$. That is, $F_{n-1}/F_n = \frac{1}{2}$. This occurs precisely once, for positive n, when n=3, and never for negative n. The only set of points is therefore that in which $OA=F_3=2$, $OB=F_4=3$, $OC=F_5=5$, $OD=F_6=8$.

Editorial note. Let $R_n = F_{n-1}/F_n$. It is well known and easily proved that $R_2 > R_4 > R_6 > \ldots R_7 > R_5 > R_3$. This shows that the n for which $R_n = \frac{1}{2}$ is unique.

Also solved by Charles R. Wall, Texas Christian University, Ft. Worth, Texas and the proposer.

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Moreover, these are the dimensions of the cuboid of unit volume, for $\varphi \times 1 \times \varphi^{-1} = 1$.

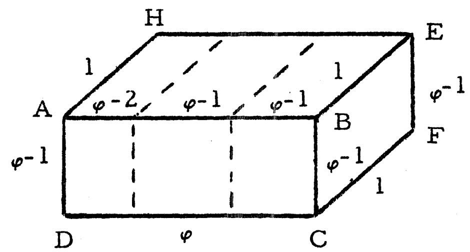

Fig. 2

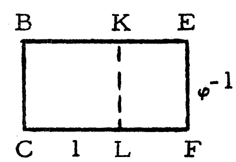

Fig. 3

Certain other properties of the Golden Cuboid may be noted.

1. It is clear from Fig. 2 that the ratios of the areas of the faces are: AE:AC:CE = $\varphi:1:\varphi^{-1}$.
2. The total surface area of the cuboid is $3(\varphi + 1 + \varphi^{-1}) = 6\varphi$
3. Four of the six faces of the cuboid are Gold Rectangles, e.g., CE (Fig. 3)
4. Each of the four diagonals of the cuboid is inclined to the base at an angle of 30°.
5. The ratio of the area of the sphere circumscribing the cuboid to that of the cuboid is $2\pi:3\varphi$.

One further point is of interest.

6. It is well known that, if a square CK is cut off from the Golden Rectangle CE (Fig. 3), the sides of the remaining rectangle LE are also in the ratio $\varphi:1$. And of course the dissection may be repeated until the rectangle size approaches that of a point, which is the intersection of BF and KE.

It is not so well known that, if two cuboids of square cross section $(\varphi^{-1} \times \varphi^{-1})$ are cut from the Golden Cuboid (broken lines, Fig. 2), the edge lengths of the remaining cuboid are in the same ratio as those of the original cuboid, viz., $1:\varphi^{-1}:\varphi^{-2} = \varphi:1:\varphi^{-1}$, so that this also is a Golden Cuboid, φ^{-3} times the size of the original.

The repetition of the decapitation process will lead to an indefinitely small Golden Cuboid located about a fixed point. The location of this point is left as an exercise to the reader.

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