ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-46 *Proposed by F.D. Parker, SUNY at Buffalo, Buffalo, New York*

Prove

$$D_n = |a_{ij}| = (-1)^n K ,$$

where $a_{ij} = F^4_{n+i+j-2}$ $(i, j = 1, 2, 3, 4, 5)$ and find the value of K.

H-47 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} L_n x^n = \frac{\psi_k(x)}{(1-x-x^2)^k} ,$$

where

$$\psi_k(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} L_r x^r .$$

H-48 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Solve the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + m^n ,$$

where C_1 and C_2 are arbitrary and m is a fixed positive integer.

H-49 *Proposed by C.R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that, for $n > 0$,

$$2^n F_{n+1} = \sum_{m=0}^{n} \frac{5^{[m/2]} n^{(m)}}{m!}$$

where $[x]$ denotes the integral part of x, and $x^{(n)} = x(x-1)\ldots(x-n+1)$.

H-50 *Proposed by Ralph Greenberg, Philadelphia, Pa. and H. Winthrop, University of South Florida, Tampa, Florida*

Show

$$\sum_{n_1+n_2+n_3+\ldots+n_i = n} \Pi\, n_i = F_{2n},$$

where the sum is taken over all partitions of n into positive integers and the order of distinct summands is considered.

H-51 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California and L. Carlitz, Duke University, Durham, N.C.*

Show that if

$$\text{(i)} \qquad \frac{xt}{1-(2-x)t+(1-x-x^2)t^2} = \sum_{k=1}^{\infty} Q_k(x)t^k$$

and

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \binom{n+k-1}{n} F_n x^n = \frac{\phi_k(x)}{(1-x-x^2)^k}$$

that

$$\phi_k(x) = \sum_{r=0}^{k} (-1)^{r+1} \binom{k}{r} F_r x^r = Q_k(x)$$

See also H-47.

LOG OF THE GOLDEN MEAN

H-29 *Proposed by Brother U. Alfred, St. Mary's College, California*

Find the value of a satisfying the relation

$$n^n + (n + a)^n = (n + 2a)^n$$

in the limit as n approaches infinity.

Solution by George Ledin, Jr., San Francisco, Calif.

Since $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$, then dividing

$$(n+a)^n + n^n = (n+2a)^n$$

through by $n^n \neq 0$ yields

$$(1+\frac{a}{n})^n + 1 = (1+\frac{2a}{n})^n,$$

which upon passing to the limit on n, gives the equation

$$e^a + 1 = e^{2a}$$

whose positive solution is $a = \ln \frac{1 + \sqrt{5}}{2} = \ln\phi$, the log of the Golden Mean.

Also solved by R. Weinshenk, Sunnyvale, California, J.L. Brown, Jr., State College, Pa., Raymond Whitney, Lock Haven, Pa., Zvi Dresner, *and the proposer.*

MORE DIOPHANTUS AND FIBONACCI

H-30 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Find all non-zero integral solutions to the two Diophantine equations,

(a) $$X^2 + XY + X - Y^2 = 0$$

(b) $$X^2 - XY - X - Y^2 = 0 \quad .$$

Report by the proposer

All solutions of $X^2 + XY + X - Y^2 = 0$ are

$$X = F_{2n}^2$$

$$Y = F_{2n}F_{2n+1}$$

All solutions of $X^2 - XY - X - Y^2 = 0$ are

$$X = F_{2n+1}^2$$

$$Y = F_{2n}F_{2n+1}$$

All solutions of $X^2 + XY - X - Y^2 = 0$ are

$$X = F_{2n+1}^2$$

$$Y = F_{2n+1}F_{2n+2}$$

All solutions to $X^2 - XY + X - Y^2 = 0$ are

$$X = F_{2n+2}^2$$

$$Y = F_{2n+1}F_{2n+2}.$$

The "only if" portion of the report was incomplete. The Editor awaits further comments from our readers.

UNIMODULAR BILINEAR TRANSFORMATIONS

H-31 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Prove the following:

Theorem: Let a, b, c, d be integers satisfying $a > 0$, $d > 0$ and $ad - bc = 1$, and let the roots of $\lambda^2 - \lambda - 1 = 0$ be the fixed points of

$$W = \frac{az + b}{cz + d}.$$

Then it is necessary and sufficient for all integral $n \neq 0$, that $a = F_{2n+1}$, $b = c = F_{2n}$, and $d = F_{2n-1}$, where F_n is the n^{th} Fibonacci number. ($F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all integral n.)

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Since the equation $\lambda^2 - \lambda - 1 = 0$ has two distinct roots $\lambda_1 = \frac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{5}}{2}$, we note that $c \neq 0$. From the fixed point conditions,

$$\lambda_1 = \frac{a\lambda_1 + b}{c\lambda_1 + d} \quad \text{and} \quad \lambda_2 = \frac{a\lambda_2 + b}{c\lambda_2 + d},$$

it is simple to derive the following necessary conditions:

$$a = c + d$$

$$b = c \ .$$

Conversely, if $b \neq 0$, $b = c$ and $a = c+d$, then the transformation becomes

$$w = \frac{az + b}{bz + (a-b)} \ ,$$

and the equation for the fixed points of this transformation is

$$z^2 - z - 1 = 0 \ ,$$

so that λ_1 and λ_2 are the fixed points.

We have thus shown that the bilinear transformation $w = \frac{az+b}{cz+d}$ has fixed points $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ <u>if and only if</u> $c \neq 0$, $b = c$ and $a = c+d$.

Substituting these latter conditions into the condition $ad - bc = 1$, we obtain the following diophantine equation relating c and d:

$$(*) \qquad c^2 - d^2 - cd + 1 = 0 \ .$$

Let (c, d) be an <u>arbitrary</u> pair of <u>positive</u> integers which satisfy (*). Then, it is clear that $c \geq d > 0$. It is easily verified that $(c-d, 2d-c)$ is also an integer solution pair for (*) with first term ≥ 0 and second term > 0. [If $2d-c < 0$, then $0 < d < \frac{c}{2}$ and $c^2 - d^2 - cd + 1 > c^2 - \frac{c^2}{4} - \frac{c^2}{2} + 1 = \frac{c^2}{4} + 1 > 1$, contradicting the fact that $c^2 - d^2 - cd + 1 = 0$.] If the first term $c-d$ is actually > 0, then we may form another solution $(2c-3d, 5d-3c)$ in the same manner and the new solution will again have a non-negative first term and positive second term. After n such iterations (assuming positive first terms), we arrive at the solution $(F_{2n-1}c - F_{2n}d, F_{2n+1}d - F_{2n}c)$. Now, consider the first terms of the solution pairs thus generated. For any n such that the first term is positive, we may construct an $n+1^{\underline{st}}$ solution which either has a positive first term or has a first term of zero. Also note the first term of each successive solution is <u>smaller</u> than the first term of the preceding solution. It is clear that our construction process must lead, in a finite number of steps, to a solution with first term 0, namely the solution (0, 1). For it not, we could produce by the foregoing process an arbitrarily large number of solution pairs (c_d, d_n) in positive integers with $c > c_1 > c_2 > c_3 \ldots.$ and $0 < d_n \leq c_n$ for each n. This infinite descent is obviously impossible; hence, there exists an integer $k > 0$ such that the solution pair $(F_{2k-1}c - F_{2k}d, F_{2k+1}d - F_{2k}c)$ is identically the pair (0, 1). We have, therefore,

$$F_{2k-1}c - F_{2k}d = 0$$

$$F_{2k+1}d - F_{2k}c = 1 ,$$

from which $c = F_{2k}$, $d = F_{2k-1}$ and $a = c+d = F_{2k+1}$. This shows the necessity of the condition that the coefficients are Fibonacci numbers of a certain form; the sufficiency follows directly using the identity $F_{2k-1}F_{2k+1} - F_{2k}^2 = 1$. This proves the stated theorem and also shows that $c = F_{2k}$ and $d = F_{2k-1}$ for $k = 1, 2, 3, \ldots$ constitute all possible solutions in positive integers of the diophantine equation $c^2 - d^2 - cd + 1 = 0$.

The reader is directed to an application of the result of H-31 in S. L. Basin's "The Appearance of Fibonacci Numbers and the Q-Matrix in Electrical Network Theory" Mathematics Magazine Volume 3b No. 2 March 1962, pp. 84-97 (see specifically Theorem 1, page 94). This theorem was first proved in an unpublished paper "The Many Facets of the Fibonacci Numbers" by V. E. Hoggatt, Jr., and Charles H. King. Also solved by Zvi Dresner.

NO FIBONACCI TRIANGLES

H-32 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, N.J.*

Prove the following:

Given a positive integer n, if there exist m line segments L_i having lengths a_i, $1 \leq a_i \leq n$, for all $1 \leq i \leq m$, such that no three L_i can be used to form a non-degenerate triangle then $F_m \leq n$, where F_m is the m^{th} Fibonacci number.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

By hypothesis, $a_1 \geq 1 = F_1$ and $a_2 \geq 1 = F_2$. Since L_1, L_2 and L_3 do not form a non-degenerate triangle, we must have (assuming the L_i have been reordered, if necessary, so that $a_1 \le a_2 \le a_3 \le \ldots \le a_m$)

$$a_3 > a_1 + a_2 \ge F_1 + F_2 = F_3 \quad .$$

Similarly, L_2, L_3 and L_4 do not form a non-degenerate triangle so that

$$a_4 > a_3 + a_2 > F_3 + F_2 = F_4 \quad .$$

Proceeding inductively in this fashion, we conclude $a_m > F_m$, and the desired result follows (actually with strict inequality) from $n \ge a_m$.

Also solved by the proposer and Zvi Dresner.

LUCAS PRIMALITY

H-33 *Proposed by Malcolm Tallman, Brooklyn, N.Y.*

If a Lucas number is a prime number and its subscript is composite, then the subscript must be of the form 2^m, $m \ge 2$.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Assume L_n is prime and has a composite subscript n. Then $n = (2r-1)\cdot 2^m$ for some $m \ge 0$ and some $r \ge 1$. It is well-known (see e.g. equation (6) of "A Note on Fibonacci Numbers" by L. Carlitz, this Quarterly, Vol. 2, No. 1, p. 15) that $L_k \mid L_{(2r-1)k}$ if $r > 1$ and hence

$$L_{2^m} \mid L_{(2r-1)2^m} \qquad \text{if} \quad r > 1 \quad .$$

Since $L_{(2r-1)2^m}$ is prime by hypothesis, we conclude $r = 1$. (The alternative $m = 0$ would force n to be a prime contrary to hypothesis). Thus $n = 2^m$ and m must be ≥ 2 in order for n to be composite.

Also solved by the proposer and Zvi Dresner.

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