ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-27 *Proposed by D.C. Cross, Exeter, England*

Corrected and restated from Vol. 1, No. 4: The Chebyshev Polynomials $P_n(x)$ are defined by $P_n(x) = \cos(n \operatorname{Arccos} x)$. Letting $\phi = \operatorname{Arccos} x$, we have

$$\cos \phi = x = P_1(x),$$

$$\cos (2\phi) = 2\cos^2 \phi - 1 = 2x^2 - 1 = P_2(x),$$

$$\cos (3\phi) = 4\cos^3 \phi - 3\cos \phi = 4x^3 - 3x = P_3(x),$$

$$\cos (4\phi) = 8\cos^4 \phi - 8\cos^2 \phi + 1 = 8x^4 - 8x^2 + 1 = P_4(x), \text{ etc.}$$

It is well known that

$$P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x) .$$

Show that

$$P_n(x) = \sum_{j=0}^{m} B_{jn} x^{n-2j}$$

where

$$m = [n/2] ,$$

the greatest integer not exceeding $n/2$, and

(1) $B_{on} = 2^{n-1}$

(2) $B_{j+1, n+1} = 2B_{j+1, n} - B_{j, n-1}$

(3) If $S_n = |B_{on}| + |B_{1n}| + \ldots + |B_{mn}|$, then $S_{n+2} = 2S_{n+1} + S_n$.

B-52 *Proposed by Verner E. Hoggatt, Jr., San Jose State Collete, San Jose, Calif.*

Show that $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, where F_n is the n-th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

B-53 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Show that

$$(2n - 1)F_1^2 + (2n - 2)F_2^2 + \ldots + F_{2n-1}^2 = F_{2n}^2 \quad .$$

B-54 *Proposed by C.A. Church, Jr., Duke University, Durham, N. Carolina*

Show that the n-th order determinant

$$f(n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & & 0 & 0 \\ -1 & a_2 & 1 & 0 & & 0 & 0 \\ 0 & -1 & a_3 & 1 & & 0 & 0 \\ 0 & 0 & -1 & a_4 & \ldots & 0 & 0 \\ \ldots \\ \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & a_n \end{vmatrix}$$

satisfies the recurrence $f(n) = a_n f(n-1) + f(n-2)$ for $n > 2$.

B-55 *From a proposal by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $x^n - xF_n - F_{n-1} = 0$ has no solution greater than a, where $a = (1 + \sqrt{5})/2$, F_n is the n-th Fibonacci number, and $n > 1$.

B-56 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let F_n be the n-th Fibonacci number. Let $x_0 \geq 0$ and define $x_1, x_2, \ldots$ by $x_{k+1} = f(x_k)$ where

$$f(x) = \sqrt[n]{F_{n-1} + xF_n} .$$

For $n > 1$, prove that the limit of x_k as k goes to infinity exists and find the limit. (See B-43 and B-54.)

B-57 *Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, Calif.*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number respectively. Prove that

$$(F_{4n}/n)^n > L_2L_6L_{10} \cdots L_{4n-2}$$

for all integers $n > 2$.

SOLUTIONS

RECURSIVE POLYNOMIAL SEQUENCES

B-26 *Proposed by S.L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Corrected statement: Given polynomials $b_n(x)$ and $B_n(x)$ defined by

$$b_0(x) = 1, \quad B_0(x) = 1$$

$$(1) \quad b_n(x) = xB_{n-1}(x) + b_{n-1}(x) \qquad (n > 0)$$

$$(2) \quad B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x) \qquad (n > 0)$$

show that $b_n(x) = P_{2n}(x)$ and $B_n(x) = P_{2n+1}(x)$ where

$$P_m(x) = \sum_{j=0}^{[m/2]} \binom{m-j}{j} x^{[m/2] - j} ,$$

$[m/2]$ being the greatest integer not exceeding $m/2$.

Solution by Lucile Morton, Santa Clara, California

We see that both $b_n(x)$ and $B_n(x)$ satisfy

(3) $$u_{n+2}(x) = (x+2)u_{n+1}(x) - u_n(x) \quad (n > 0),$$

as follows: Subtracting corresponding sides of (1) from those of (2), we have $B_n(x) - b_n(x) = B_{n-1}(x)$. Then $b_n(x) = B_n(x) - B_{n-1}(x)$ and it follows from (2) that

$$B_n(x) = (x+1)B_{n-1}(x) + B_{n-1}(x) - B_{n-2}(x) = (x+2)B_{n-1}(x) - B_{n-2}(x).$$

Hence $B_n(x)$ satisfies (3). Then so does $B_{n-1}(x)$ and the difference $B_n(x) - B_{n-1}(x) = b_n(x)$.

A lengthy but not difficult induction confirms that $P_{2n}(x)$ and $P_{2n+1}(x)$ both satisfy (3). Since they have the same initial values as $b_n(x)$ and $B_n(x)$ respectively, this establishes the desired result.

Also solved by the proposer.

ARITHMETIC PROGRESSIONS

B-38 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

Characterize simply all the sequences c_n satisfying

$$c_{n+2} = 2c_{n+1} - c_n .$$

Solution by J.L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

From

$$c_{n+2} - c_{n+1} = c_{n+1} - c_n ,$$

it is clear that the differences between successive terms must be a constant independent of n. Letting c_1 and c_2 be two arbitrary specified initial values, we obtain

$$c_n = c_2 + (n-2)(c_2 - c_1) .$$

Also solved by George Ledin, Jr., University of California, Berkeley, Calif; Douglas Lind, Falls Church, Virginia; Raymond Whitney, Pennsylvania State University, Hazleton, Pennsylvania; J.A.H. Hunter, Toronto, Ontario, Canada; Dermott A. Breault, Sylvania–A.R.L., Waltham, Mass.; and the proposer.

BOUNDS FOR FIBONACCI NUMBERS

B-39 *Proposed by John Allen Fuchs, University of Santa Clara, Santa Clara, California*

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$. Prove that

$$F_{n+2} < 2^n \text{ for } n \geq 3 .$$

Solution by Brian Scott, Ripon, Wisconsin

The solution is by induction on n. $F_{3+2} = F_5 = 5 < 8 = 2^3$ and $F_{4+2} = F_6 = 8 < 16 = 2^4$. Assume as the induction hypothesis that $F_{(n-2)+2} < 2^{n-2}$ and $F_{(n-1)+2} < 2^{n-1}$. Then $F_{n+2} = F_{(n-1)+2} + F_{(n-2)+2} < 2^{n-1} + 2^{n-2} = 2^{n-2}(2+1) < 2^{n-2} \cdot 2^2 = 2^n$. Therefore $F_{n+2} < 2^n$ for all $n \geq 3$.

Also solved by Gladwin E. Bartel, University of Wisconsin, Madison, Wisconsin; Dermott A. Breault, Sylvania–A.R.L., Waltham, Massachusetts; John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; George Ledin, Jr., University of California, Berkeley, California; Douglas Lind, Falls Church, Virginia; Howard Walton, Yorktown H.S., Arlington, Virginia; John Wessner, Melbourne H.S., Melbourne, Florida; Raymond Whitney, Pennsylvania State University, Hazelton, Pennsylvania; Charles Ziegenfus, Madison College, Harrisonburg, Virginia; and the proposer.

Lind mentioned the related $F_n < (7/4)^n$ on page 7 of Topics in Number Theory by W. J. LeVeque and $a^{n-1} < F_n < a^n$, where

$$a = (1 + \sqrt{5})/2 ,$$

on page 93 of An Introduction to the Theory of Numbers, by Niven and Zuckerman. Ziegenfus mentioned similar problems in LeVeque's Elementary Theory of Numbers.

A SUMMATION FORMULA

B-40 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

If H_n is the n-th term of the generalized Fibonacci sequence, i.e., $H_1 = p$, $H_2 = p+q$, $H_{n+2} = H_{n+1} + H_n$ for $n \geq 1$, show that

$$\sum_{1}^{n} k H_k = (n+1) H_{n+2} - H_{n+4} + 2p + q .$$

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pensylvania

$H_1 = 2H_3 - H_5 + 2p + q$, so that the assertion is true for $n = 1$. Assume as an induction hypothesis that the result has been proved for all n satisfying $1 \le n \le m$, where $m \ge 1$. We will show the result must necessarily hold for $m + 1$.

$$\sum_1^{m+1} k H_k = (m+1) H_{m+1} - \sum_1^{m} k H_k$$

$$= (m+1) H_{m+1} + (m+1) H_{m+2} - H_{m+4} + 2p + q$$

$$= (m+1) H_{m+3} - H_{m+4} + 2p + q$$

$$= (m+2) H_{m+3} - (H_{m+4} + H_{m+3}) + 2p + q$$

$$= (m+2) H_{m+3} - H_{m+5} + 2p + q \ .$$

Hence, the assertion holds for $n = m+1$ and the proof is completed by the usual inductive argument.

Also solved by Dermott A. Breault, Sylvania–A.R.L., Waltham, Massachusetts; Douglas Lind, Falls Church, Virginia; George Ledin, Jr., University of California, Berkeley, California; Howard Walton, Yorktown H.S., Arlington, Virginia; and the proposer.

AN IMPOSSIBLE CONDITION

B-41 *Proposed by David L. Silverman, Beverley Hills, California*

Do there exist four distinct positive Fibonacci numbers in arithmetic progression?

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

No. For, assume $F_i < F_j < F_h < F_k$ are in arithmetic progression, so that $F_j - F_i = d = F_k - F_h$. Then

$$d = F_j - F_i < F_j$$

while

$$d = F_k - F_h \ge F_k - F_{k-1} = F_{k-2} \ge F_j \ ,$$

since $k \geq j+2$. This is a contradiction, so that four distinct positive Fibonacci numbers cannot be in arithmetic progression.

Also solved by Brian Scott, Ripon, Wisconsin and the proposer

F_{n+1} IN TERMS OF F_n

B-42 *Proposed by S.L. Basin, Sylvania Electronics Systems, Mountain View, California*

Express the (n + 1)-st Fibonacci number F_{n+1} as a function of F_n. Also solve the same problem for L_n.

Solution by H.H. Ferns, University of Victoria, Victoria, B.C., Canada

The following three identities are readily proved by applying Binet's formula.

(1) $$2F_{n+1} = F_n + L_n$$

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n$$

(3) $$2L_{n+1} = 5F_n + L_n$$

Eliminating L_n from (1) and (2) gives

$$F_{n+1} = \frac{F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2} ,$$

Eliminating F_n from (2) and (3) gives

$$L_{n+1} = \frac{L_n + \sqrt{5}\sqrt{L_n^2 - 4(-1)^n}}{2} ,$$

Also solved by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Douglas Lind, Falls Church, Virginia; and the proposer

ITERATION FOR THE GOLDEN MEAN

B-43 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

(1) Let $x_0 \geq 0$ and define a sequence x_k by $x_{k+1} = f(x_k)$ for $k \geq 0$, where $f(x) = \sqrt{1 + x}$. Find the limit of x_k as $k \to \infty$.

(2) Solve the same problem for $f(x) = \sqrt[3]{1 + 2x}$.

(3) Solve the same problem for $f(x) = \sqrt[4]{2 + 3x}$.

(4) Generalize.

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

(1) If
$$\lim_{k \to \infty} x_k$$
exists, call it x. Then we have, since f(x) is continuous,
$$\lim_{k \to \infty} x_{k+1} = x = \lim_{k \to \infty} \sqrt{1 + x_k} = \sqrt{1 + x}, \text{ or } x^2 = 1 + x \quad ,$$
yielding
$$x = \frac{1 + \sqrt{5}}{2}$$
as the unique positive solution for the limit.

(2) The same process yields
$$x^3 = 1 + 2x, \text{ or } x^3 - 2x - 1 = (x+1)(x^2-x-1) = 0 \quad .$$
Again, there is only one positive solution, namely $x = \frac{1 + \sqrt{5}}{2}$.

(3) Similarly, the equation $x^4 = 2 + 3x$ or $x^4 - 3x - 2 = 0$ clearly has only one positive root since the quantity $x^4 - 3x - 2$ is negative for $0 \le x \le 1$ and is monotonic increasing for $x > 1$. This unique positive root, which is easily verified to be $\frac{1 + \sqrt{5}}{2}$, is the required limit.

(4) If f(x) is continuous and is such that
$$\lim_{k \to \infty} x_k = L$$
exists when $x_{k+1} = f(x_k)$, then the limit L is a solution of the equation $x = f(x)$. If, further, f(x) is positive for all x, then x must be non-negative. Note: The solution above does not prove that $x^n = F_{n-1} + F_n x$ has no solution with $x > (1 + \sqrt{5})/2$. This is left to the reader as B-55 below.

Also solved by George Ledin, Jr., San Francisco, Calif.; Douglas Lind, Falls Church, Virginia; and the proposer

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