APPLICATION OF FIBONACCI NUMBERS TO SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

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A casual glance at a system of linear equations such as

$$2584x + 4181y = 20$$

$$4181x + 6765y = 21$$

might lead one to think that it is unlikely that the solution could consist of integers. However, a closer look by regular readers of this journal will reveal that the coefficients of x and y in both equations are Fibonacci numbers and that if the general notation of Fibonacci numbers in which F_n denotes the nth term of the sequence 1, 1, 2, 3, ..., is used, then the equations above turn out to be the special case $n = 20$ of the general form of the system of equations:

$$(F_{n-2})x + (F_{n-1})y = n \ , \tag{1}$$

$$(F_{n-1})x + (F_n)y = n + 1. \tag{2}$$

The solution to such a system of equations is

$$\frac{n(F_n) - (n+1)(F_{n-1})}{(F_n)(F_{n-2}) - (F_{n-1})(F_{n-1})} = x, \quad \text{and} \tag{3}$$

$$\frac{n(F_{n-1}) - (n+1)(F_{n-2})}{(F_{n-1})(F_{n-1}) - (F_{n-2})(F_n)} = y \quad . \tag{4}$$

The denominators of the fractions in equations (3) and (4) have the interesting property

$$(F_n)(F_{n-2}) - (F_{n-1})(F_{n-1}) = \begin{matrix} +1 \text{ if } n \text{ is odd,} \\ -1 \text{ if } n \text{ is even} \end{matrix} \ . \tag{5}$$

$$(F_{n-1})(F_{n-1}) - (F_{n-2})(F_n) = \begin{matrix} -1 \text{ if } n \text{ is odd,} \\ +1 \text{ if } n \text{ is even} \end{matrix} \ . \tag{6}$$

It may be noted that statements (5) and (6) are equivalent.

Equations (5) and (6) permit one to write equations (3) and (4) in more convenient form:

(7a) $x = n(F_n) - (n+1)(F_{n-1})$ if n is odd,

(7b) $x = (n+1)(F_{n-1}) - n(F_n)$ if n is even.

(8a) $y = (n+1)(F_{n-2}) - n(F_{n-1})$ if n is odd,

(8b) $y = n(F_{n-1}) - (n+1)(F_{n-2})$ if n is even.

Since the numbers n, $n + 1$, F_n, F_{n-1}, F_{n-2} are integers, and since the set of integers is closed under the operations of addition, subtraction, and multiplication, it follows that all solutions to the system of linear equations represented in equations (1) and (2) are integers.

In the accompanying table, the symbols ξ_n and ξ_{n+1} denote equations of the form (1) and (2) respectively and the solution to such a system is symbolized as $\xi_n \cap \xi_{n+1}$. It may be noted that any equation* ξ_n is the sum of the two equations* above it in the table, provided that $n \geq 3$. It may also be noted that the coefficients of both x and y occur in the well known Fibonacci sequence.

Equations (5) and (6) were derived intuitively by the writer who suggests that readers attempt formal proofs of them. The suggestion is also made one might consider investigating the sequences of numbers which constitute the solutions to such systems.

$$X_{2n+1} = (2n+1)\ F_{2n+1} - (2n+2)\ F_{2n}$$

$$= (2n+1)\ F_{2n-1} - F_{2n}$$

$$X_{2n+2} = (2n+1)\ F_{2n-1} - 2nF_{2n} \qquad \text{etc.}$$

$$= (2n+1)\ F_{2n-1} - 2n\ (F_{2n-1} + F_{2n-2})$$

$$= F_{2n-1} - 2n\ F_{2n-2}$$

*Left sides of these equations only.

n	ξ_n	$\xi_n \cap \xi_{n+1}$
1	$x = 1$	(1, 2)
2	$y = 2$	(1, 2)
3	$x + y = 3$	(2, 1)
4	$x + 2y = 4$	(-2, 3)
5	$2x + 3y = 5$	(7, -3)
6	$3x + 5y = 6$	(-13, 9)
7	$5x + 8y = 7$	(27, -16)
8	$8x + 13y = 8$	(-51, 32)
9	$13x + 21y = 9$	(96, -56)
10	$21x + 34y = 10$	(-176, 109)
11	$34x + 55y = 11$	(319, -197)
12	$55x + 89y = 12$	(-571, 353)
13	$89x + 144y = 13$	(1013, -626)
14	$144x + 233y = 14$	(-1783, 1102)
15	$233x + 377y = 15$	(3118, -1927)
16	$377x + 610y = 16$	(-5422, 3351)
17	$610x + 987y = 17$	(9383, -5799)
18	$987x + 1597y = 18$	(-16169, 9993)
19	$1597x + 2584y = 19$	(27759, -17156)
20	$2584x + 4181y = 20$	(-47499, 29356)
21	$4181x + 6765y = 21$	....
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