ON THE CONVERGENCE OF ITERATED EXPONENTIATION—III

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(Submitted May 1979)

The present paper can be considered as an extension of two previous papers in which the properties of the following function were discussed (see [1] and [2]):

(1) $$F(x, y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}}},$$

where an infinite number of exponentiations is understood. Equation (1) is the function specifically studied in [2], whereas in [1] we considered the simpler function

(2) $$f(x) \equiv F(x, x),$$

i.e., the case of Eq. (1) where $x = y$. For both Eqs. (1) and (2), the ordering of the exponentiations is important, and for Eq. (1) and throughout this paper, we mean a bracketing order "from the top down," i.e., x raised to the power y, followed by y raised to the power x^y, and then x raised to the power $y^{(x^y)}$, and so on, all the way down to the x which is at the lowest position of the "ladder."

In the present paper, we study the properties of a function which is obtained by forming an infinite sequence of roots. We have restricted ourselves to a single (positive) variable x, i.e., the analogue of Eq. (2). We will call this function $\phi(x)$, and it is defined as follows:

(3) $$\phi(x) = \sqrt[\sqrt[\cdot^{\cdot^{\sqrt[x]{x}}}]{x}]{x},$$

where an infinite number of roots is understood. The bracketing is again from "the top down," i.e., we mean $\sqrt[x]{x}$, followed by the $\sqrt[x]{x}$-th root of x, which can be written as $\xi(x)$, followed by the root $\sqrt[\xi]{x}$, and so on, down to the lowest x in the "ladder."

From Eq. (3), it can be see that we have:

(4) $$\phi(x) = x^{\frac{1}{\phi(x)}} = \sqrt[\phi(x)]{x},$$

provided that the sequence (3) has a nontrivial limit. From Eq. (4), we obtain the equation:

(5) $$\phi(x)^{\phi(x)} = x.$$

Values of $\phi(x)$ were calculated by means of a simple program embodying the sequential operations of Eq. (3) on a Hewlett-Packard calculator. In this manner, we have obtained the graph of Figure 1, in which $\phi(x)$ is shown as a function of x. We note that for $x < e^{-1/e}$, i.e., $x < 0.692200\ldots$, $\phi(x) = 0$, and at $x = e^{-1/e}$, $\phi(x)$ has the value $1/e = 0.36788$. Indeed, for $\phi = e^{-1}$, Eq. (5) gives

$$\phi^\phi = e^{-1(1/e)} = e^{-1/e} = x.$$

The reason for the abrupt decrease of $\phi(x)$ to zero below $x = e^{-1/e}$ is illustrated

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in Figure 2, in which we have plotted ϕ^ϕ as a function of ϕ. It can be seen that ϕ^ϕ has a minimum value of $e^{-1/e}$ which is attained at $\phi = 1/e$. Indeed, the derivative $d\phi^\phi/d\phi$ is zero at this point, as can be seen from the following equation:

$$\frac{d\phi^\phi}{d\phi} = \frac{d\exp(\phi\log\phi)}{d\phi} = \phi^\phi(\log\phi + 1). \tag{6}$$

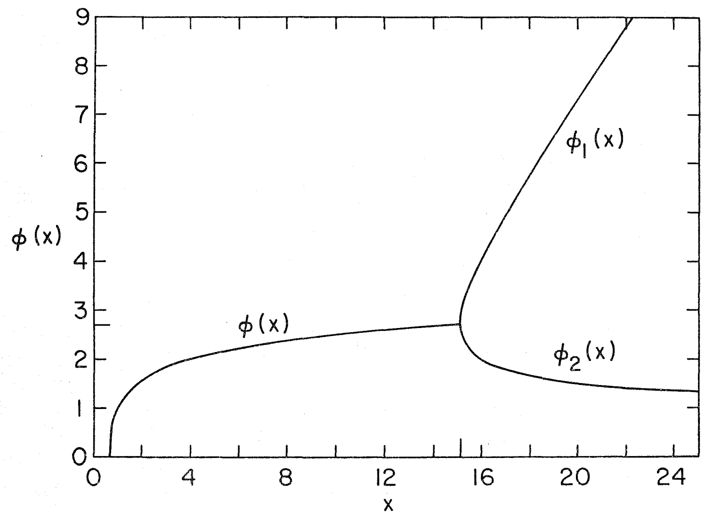

Fig. 1. *The curve of the function $\phi(x)$ as a function of x. For $x < e^{-1/e} = 0.6922$, $\phi(x) = 0$. At $x = e^{-1/e}$, $\phi(x) = 1/e \cong 0.36788$, so that $\phi(x)$ has an abrupt discontinuity at $x = e^{-1/e}$. For $x > e^e = 15.1542$, the sequence $\phi(x)$ defined by Eq. (3) converges to two different values $\phi_1(x)$ and $\phi_2(x)$, depending on whether the number n of x's is odd or even, respectively. This property can be called "dual convergence" and has been described previously in [1-3].*

Thus, for $x < e^{-1/e}$, Eq. (5) has no solution with $\phi(x) > 0$. At $\phi = 0$, the derivative $d\phi^\phi/d\phi \to -\infty$, since $\log\phi \to -\infty$. We also note from Figure 2 that for $e^{-1/e} < \phi^\phi < 1$, there are two values of ϕ for a given value of ϕ^ϕ. Thus, we can divide the curve of Figure 2 into two branches, the one to the left of $\phi = 1/e$, and the other to the right of $\phi = 1/e$. The branch to the right of $\phi = 1/e$, i.e., the branch with $\phi > 1/e$, gives the value of ϕ for a given x, as obtained from Eq. (3). The meaning of the other (left) branch will be discussed below. We note that for $\phi > 1$, there is a unique value of ϕ for a given $\phi^\phi = x$, as shown in Figure 2.

Returning now to Figure 1, we note that for $x > e^e = 15.1542\ldots$, we have a dual convergence of Eq. (3), namely a convergence to two values $\phi_1(x)$ and $\phi_2(x)$ depending upon whether the number n of x's in Eq. (3) is odd or even. This property of dual convergence has been discussed previously in connection with the function $f(x) = F(x, x)$ of [1] for $x < e^{-e} = 0.06599$. The concept of dual convergence was actually introduced in an earlier paper by the authors [3] which was circulated as a Brookhaven Informal Report [4].

At the point $x = e^e$, $\phi(x)$ has the unique value $\phi(x) = e$, which is marked on the ordinate axis of Figure 1. For very large x, it is easy to show that $\phi_1(x)$ approaches x, whereas $\phi_2(x)$ approaches 1. In order to illustrate this property, we consider the choice $x = 10{,}000$. Now $\sqrt[x]{x} = 10{,}000^{0.0001} = 1.000922$, and the next step calls for the calculation of

$$10{,}000^{1/1.000922} = 9915.53,$$

followed by

$$10{,}000^{1/9915.53} = 1.000929.$$

The actual values to which the infinite sequence of Eq. (3) converges for $x = 10{,}000$ are:

$$\phi_1(x) = 9914.85 \quad \text{and} \quad \phi_2(x) = 1.0009294.$$

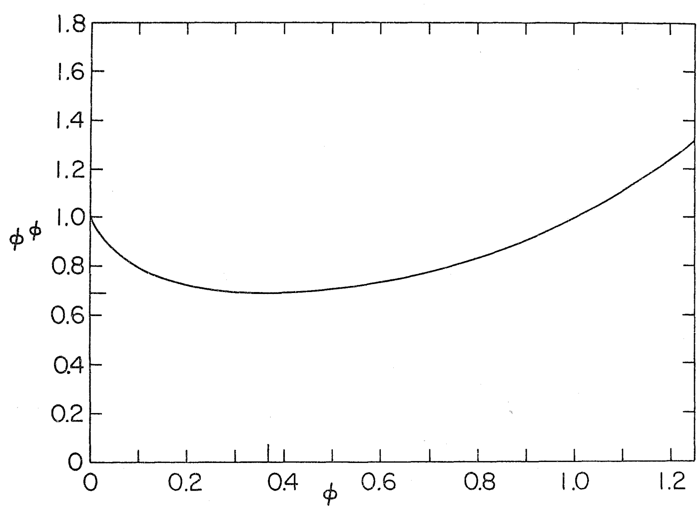

Fig. 2. The function ϕ^ϕ as a function of ϕ for ϕ in the region $0 < \phi < 1.25$. This function is of interest in connection with Eq. (5), according to which $\phi^\phi = x$. We note that the minimum value of ϕ^ϕ is $e^{-1/e} = 0.6922$ and is attained at $\phi = 1/e$. Thus, for $x < 1$, the function ϕ^ϕ can be considered as having two branches, the one to the left of $\phi = 1/e$ and the one to the right of $\phi = 1/e$. The right-hand branch gives the value of ϕ as a function of $x = \phi^\phi$, e.g., for $x = 0.8$, we have $\phi(x) = 0.7395$. The left-hand branch gives the value of $N_{\min}$, as explained in the text [see Eqs. (12)-(18)]. Thus, for values of x between $e^{-1/e}$ and 1, $\phi_N(x) = \phi(x)$, provided $N \geq N_{\min}$. For $N < N_{\min}$, $\phi_N(x) = 0$. As an example, $N_{\min}(x = 0.8) = 0.09465$.

Obviously, from the definition of $\phi_1(x)$ and $\phi_2(x)$, we have the relations:

$$\phi_1(x)^{\phi_2(x)} = \phi_2(x)^{\phi_1(x)} = x \tag{7}$$

for $x > e^e$. Incidentally, the equation $\phi(x)^{\phi(x)} = x$ continues to have a solution for $x > e^e$, but this solution does not give the values of $\phi(x)$ to which the sequence (3) approaches by dual convergence. As examples of values of $\phi_1(x)$ and $\phi_2(x)$ for $x > e^e$, we may cite:

$$\text{for } x = 20: \quad \phi_1(20) = 7.2802, \quad \phi_2(20) = 1.50907;$$
$$\text{for } x = 100: \quad \phi_1(100) = 76.379, \quad \phi_2(100) = 1.06215.$$

The occurrence of $x = e^{-1/e}$ and $x = e^e$ as limiting values for $\phi(x)$ and the similar occurrence of $x = e^{1/e}$ and $x = e^{-e}$ as limiting values for $f(x)$ suggests a recriprocal relationship between the functions $\phi(x)$ and $f(x)$. This conjecture is strengthened by the fact that the values of $f(x)$ and $\phi(x)$ at corresponding points are the reciprocals of one another. Thus, we have:

$$\phi(x = e^{-1/e}) = 1/e, \qquad f(x = e^{1/e}) = e, \tag{8}$$

$$\phi(x = e^{e}) = e, \qquad f(x = e^{-e}) = 1/e. \tag{9}$$

We now prove the following relation between $\phi(x)$ and $f(x)$:

$$\phi(x) = \frac{1}{f(1/x)}. \tag{10}$$

Thus, the region of dual convergence of $\phi(x)$ for $x > e^e$ corresponds point-for-point to the region of dual convergence of $f(x)$ for $x < e^{-e}$, in which $f(x)$ has two branches $f_1(x)$ and $f_2(x)$, which approach the limiting values $f_1(x) \to x$ as $x \to 0$ for an odd number of x's in Eqs. (1) and (2), and $f_2(x) \to 1$ as $x \to 0$ for an even number of x's.

In order to prove the relation of Eq. (10), we simply note that:

$$\phi(x) = x^{\frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}}} = 1 \Bigg/ \frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}} = \left[f\left(\frac{1}{x}\right)\right]^{-1} \tag{11}$$

where the bracketing is "from the top down" the ladder, as in all of the present work. Thus, all of the arguments given for the single or dual convergence of $f(x)$ in [1] apply to the present case, provided that $x > 0$.

We now wish to consider a generalization of $\phi(x)$ to be denoted by $\phi_N(x)$, analogously to the generalization of $f(x)$ to the function $f_N(x)$ of [2]. Thus, we define $\phi_N(x)$ as follows:

$$\phi_N(x) = \sqrt{x}^{\sqrt{x}^{\cdot^{\cdot^{\cdot^{\sqrt[N]{x}}}}}}, \tag{12}$$

where N is an arbitrary positive quantity. By the same procedure as in Eq. (11), we can rewrite Eq. (12) as follows:

$$\phi_N(x) = x^{\frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{N}}}}}}} = 1 \Bigg/ \frac{1}{x}^{\frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{N}}}}}}} = \left[f_{1/N}\left(\frac{1}{x}\right)\right]^{-1} \tag{13}$$

[see Eq. (26) of 2]. For values of $x > 1$, we have $1/x < 1$, and as shown in [2, discussion following Eq. (29)], we have

$$f_{1/N}\left(\frac{1}{x}\right) = f\left(\frac{1}{x}\right) \tag{14}$$

for all values of N, and correspondingly: $\phi_N(x) = \phi(x)$. This statement applies both to the region $1 < x \leq e^e$, where $\phi(x)$ is single-valued, and to the region $x > e^e$, where we have dual convergence. In this case:

$$\phi_{1,N}(x) = \phi_1(x) \quad \text{and} \quad \phi_{2,N}(x) = \phi_2(x).$$

The situation is different when $x < 1$. As shown above, $\phi(x)$ is nonzero only in the limited region extending from $x = e^{-1/e} = 0.6922$ to $x = 1$. The corresponding values of $1/x$ are larger than 1, and hence $f_{1/N}(1/x)$ may diverge, depending on the value of N, giving $\phi_N(x) = 0$.

It has been shown in [2] that, for the function $f_{\overline{N}}(\overline{x})$, the upper limit on $\overline{N}$ is given by the root of the equation

$$\overline{x}^f = f, \tag{15}$$

where we must choose the upper branch of the curve of f vs. $\overline{x}$, i.e., the branch for which $f > e$, which we have denoted by $f^{(2)}$ [2, see the discussion following Eq. (28)]. We can therefore write $f^{(2)} = \overline{N}$. Now in view of Eq. (13), the lower limit on N for $\phi_N(x)$ is given by $N = 1/\overline{N}$, and the value of x is given by $x = 1/\overline{x}$. Upon inserting these substitutions into Eq. (15), we obtain:

$$\left(\frac{1}{x}\right)^{1/N} = \frac{1}{N}. \tag{16}$$

Upon taking the reciprocal on both sides of this equation, we find

$$x^{1/N} = N, \tag{17}$$

whence

$$x = N^N. \tag{18}$$

This equation for N is identical to the equation for $\phi(x)$ given in Eq. (5). Since $\overline{N} > e$ by the previous argument, we find $N < 1/e$, and therefore the relation of Eq. (18) for N, i.e., $N_{\min}$ (minimum value of N) corresponds to the part of the curve of $\phi^\phi = x$ which lies to the left of the point $\phi = 1/e$. Thus, the values of Figure 2 for $\phi < 1$ give both the value of $\phi(x)$ (right part of the curve) and the value of $N_{\min}(x)$ (left part of the curve), such that for $N < N_{\min}$, the function $\phi_N(x)$ of Eq. (13) is zero, even though the simple function $\phi(x)$ (with an x on top of the ladder) is convergent and nonzero, and in fact $\phi(x) \geq 1/e$.

In connection with the iterated root-taking which is implied by Eq. (3) for the function $\phi(x)$, we have considered another possible function obtained by iteration, namely:

$$R(n,\ a,\ x) \equiv \sqrt[n]{a + x\sqrt[n]{a + x\sqrt[n]{\ldots}}}\ . \tag{19}$$

Assuming the convergence of Eq. (19), we find:

$$R^n = a + xR. \tag{20}$$

For the case $n = 2$ (repeated square roots), Eq. (20) can be solved directly, with the result:

$$R(2,\ a,\ x) = \frac{x}{2} + \left(\frac{x^2}{4} + a\right)^{\frac{1}{2}}. \tag{21}$$

Also, for the special case that $a = 0$ in Eq. (19), we obtain, for arbitrary (positive) n:

$$R^n = xR, \tag{22}$$

which gives

$$R(n,\ 0,\ x) = x^{1/(n-1)}. \tag{23}$$

If, furthermore, we take $n = x$, we obtain:

$$R(x,\ 0,\ x) = x^{1/(x-1)}. \tag{24}$$

It can be easily shown that the function $R(x,\ 0,\ x)$ decreases monotonically from $\sim 1/x$ near $x = 0$ to $R = e$ at $x = 1$ and, further, to $R = 1$ as $x \to \infty$.

In Eq. (23), we note that $R(2,\ 0,\ x) = x$, i.e.,

$$x = \sqrt[2]{x\sqrt[2]{x\sqrt[2]{\ldots}}}\ . \tag{25}$$

Finally, we wish to show the connection of $R(2,\ a,\ x)$ to the continued fraction $F_c(a,\ x)$ defined as follows:

$$F_c(a,\ x) = x + \cfrac{a}{x + \cfrac{a}{x + \ldots}}. \tag{26}$$

From Eq. (26), we obtain the following equation determining the value of $F_c(a, x)$:

(27) $$F_c - x = \frac{a}{F_c},$$

whence:

(28) $$F_c^2 - xF_c - a = 0.$$

This equation is identical to the one which determines the continued square root $R(2, a, x)$, and correspondingly

(29) $$F_c(a, x) = R(2, a, x).$$

An interesting result of Eq. (28) is that in the limit that $x \to 0$, we find

(30) $$\lim_{x \to 0} F_c(a, x) = a^{\frac{1}{2}},$$

which does not seem obvious from the definition of $F_c(a, x)$ by Eq. (26).

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4. We note that the function $f(x)$ has also been considered by Perry B. Wilson, Stanford Linear Accelerator Report PEP-232 (February 1977), and by A. V. Grosse, quoted by M. Gardner, *Scientific American* 228 (May 1973):105, and by R. A. Knoebel (to be published in the *American Mathematical Monthly*).

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GENERALIZED FERMAT AND MERSENNE NUMBERS

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1. INTRODUCTION

The numbers $F_n = 1 + 2^{2^n}$ and $M_p = 2^p - 1$, where n is a nonnegative integer and p is a prime, are called Fermat and Mersenne numbers, respectively. Properties of these numbers have been studied for centuries and most of them are well known. At present, the number of known Fermat and Mersenne primes are five and twenty-seven, respectively. It is well known that if $2^n - 1 = p$, a prime, then n is a prime. It is quite easy to show that if $2^n - 1 = pq$, p and q are primes, then either n is a prime or $n = v^2$, where v is a prime. Thus

$$2^{v^2} - 1 = pq = (2^v - 1)(2^{v(v-1)} + \cdots + 2^v + 1),$$

where $2^v - 1 = p$ is a Mersenne prime. This leads to the following definition. Let k and n be positive integers. The number $L(k, n)$ is defined as follows:

$$L(k, n) = 1 + 2^n + (2^n)^2 + \cdots + (2^n)^{k-1}.$$