When $d = 3$, $a_2 = 1$, $a_3 = 1$, we obtain from (4.2) the Neumann sequence (2.4), which, as we have noted, can also be generated by Wilson's function (2.1).

Finally, we observe that

$$(4.3) \qquad \begin{cases} f_r = F_r(1, 5) = (-1)^r/5 \cdot F_{r+1}(1, 5) \\ \ell_r = F_r(1, 1) + (-1)^r/F_r(1, 1). \end{cases}$$

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A GENERALIZATION OF THE DIRICHLET PRODUCT

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1. INTRODUCTION

If f is the Dirichlet product of arithmetical functions g and h, then by definition

$$f(n) = \sum_{d|n} g(d)h(n/d).$$

In this paper we define a convolution of two arithmetical functions that generalizes the Dirichlet product. With this new convolution, which we shall refer to as the the "k-prime product," it is possible to define arithmetical functions which are analogs of certain well-known functions such as Euler's function $\phi(n)$, defined implicitly by the relation

$$(1.1) \qquad \sum_{d\delta = n} \phi(d) = n.$$

Other well-known functions to be considered in this paper include $\tau(n)$ and $\sigma(n)$ given by $\tau(n) = \Sigma 1$ and $\sigma(n) = \Sigma d$, where the summations are over the positive divisors of n. The familiar Moebius function $\mu(n)$ is defined as the multiplicative function with the evaluation $\mu(p) = -1$ and $\mu(p^e) = 0$ if $e > 1$, and satisfies the relation

$$(1.2) \qquad \sum_{d\delta = n} \mu(d) = \varepsilon(n) \equiv \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu(1) = 1$, since μ is a nonzero multiplicative function. Upon applying the Moebius inversion formula to (1.1), one obtains the simple Dirichlet product representation for ϕ,

$$(1.3) \qquad \phi(n) = \sum_{d\delta = n} \mu(d)\delta.$$

Another function which may be defined by means of the Dirichlet product is $q(n)$, the characteristic function of the set Q of square-free integers, (1.4) $$q(n) = \sum_{d\delta = n} \nu_2(d) = \sum_{d^2\delta = n} \mu(d),$$

where $\nu_2(n) = \mu(m)$ if $n = m^2$ and $\nu_2(n) = 0$ otherwise. The representations (1.3) and (1.4) are extremely useful in the development of the theory of the Euler function and the set of square-free integers.

In Section 2, we define appropriate generalizations of the concepts mentioned above and prove a generalized Meobius inversion formula (Theorem 2.4).

Included in Section 3 is a short discussion of the usefulness of the results obtained in Section 2 and an indication of the direction in which further study should be directed.

2. THE GENERALIZED PRODUCT

For each integer $k \geq 1$, let L_k represent the set of positive integers n with the property that if a prime p divides n, then p^k also divides n. A number in L_k is said to be "k-full." Let Q_k be the set of positive integers n such that each prime divisor of n has multiplicity less than k. A number in Q_k is said to be "k-free." Any positive integer n can be written uniquely in the form $n = n_1 n_2$, where $n_1 \in L_k$, $n_2 \in Q_k$, and $(n_1, n_2) = 1$. If m and n are positive integers with unique decompositions $m = m_1 m_2$ and $n = n_1 n_2$, then m and n are said to be "relatively k-prime" [notation: $(m, n)_k = 1$] provided that $(m_2, n_2) = 1$. Given arithmetical functions $f(n)$ and $g(n)$, we define the "k-prime product" of f and g (notation: $f \circ g$) as follows:

$$(f \circ g)(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = n}} f(d)g(\delta).$$

For $k = 1$, the k-prime product reduces to the Dirichlet product. The next two theorems are proved by arguments similar to those used in the case $k = 1$.

THEOREM 2.1: The k-prime product is an associative operation.

More can be said about the algebraic structure of our system. As is the case in the Dirichlet product, the arithmetic functions form a cummulation ring with unity under addition and the k-prime product.

THEOREM 2.2: If each of g and h is a multiplicative function, then $g \circ h$ is multiplicative.

We now define the generalization of the Moebius function which was mentioned earlier.

DEFINITION 2.1: Let $\mu_k(n)$ denote the multiplicative function for which $\mu_k(p^n)$ is -1, 1, 0 whenever $0 < n < k$, $k < n < 2k$, and $n \geq 2k$, respectively. Clearly, this is a valid generalization of Moebius' function, and we shall see later on that $\mu_k(n)$ plays much the same role in the development of the theory for the k-prime product as $\mu(n)$ does in the case of the Dirichlet product. In particular, we have the following two theorems.

THEOREM 2.3: $$\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d) = \varepsilon(n).$$

PROOF: For $n = 1$ the theorem is obvious. By Theorem 2.2, we need only prove the theorem for prime powers, $n = p^e$, $e > 0$. Now, if $e < k$, we have

$$\sum_{\substack{d\delta = p^e \\ (d,\delta)_k = 1}} \mu_k(d) = \mu_k(1) + \mu_k(p^e) = 1 - 1 = 0,$$

by the definition of relatively k-prime and μ_k. In the case $e \geq k$, we have

$$\sum_{\substack{d\delta = p^e \\ (d,\delta)_k = 1}} \mu_k(d) = \sum_{\substack{a=0 \\ \max(a, e-a) \geq k}}^{e} \mu_k(p^a) = \sum_{\substack{a=0 \\ \max(a, e-a) \geq k \\ a < 2k}}^{e} \mu_k(p^a)$$

by definition of μ_k. And this expression is $k - k$ or $(e - k + 1) - (e - k + 1)$, according as $e \geq 2k$ or $k \leq e < 2k$. In either case, we have the desired result.

Let $\ell(n)$ denote the arithmetical function which is identically 1.

THEOREM 2.4: If both f_1 and f_2 are arithmetical functions, then $f_1 = f_2 \circ \ell$ if and only if $f_2 = \mu_k \circ f_1$.

PROOF: If

$$f_2(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d) f_1(\delta),$$

then

$$\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} f_2(n) = \sum_{\substack{DE\delta = n \\ (DE,\delta)_k = 1 \\ (D,E)_k = 1}} \mu_k(D) f_1(E) = \sum_{\substack{DE\delta = n \\ (D,\delta)_k = 1 \\ (E,D\delta)_k = 1}} \mu_k(D) f_1(E)$$

$$= \sum_{\substack{E \mid n \\ (E,D\delta)_k = 1}} f_1(E) \sum_{\substack{D\delta = n/E \\ (D,\delta)_k = 1}} \mu_k(D).$$

The inner sum here is 1 is $n/E = 1$ and 0 otherwise, by Theorem 2.3, so the expression reduces to $f_1(n)$. The proof of the other half is similar.

It is interesting to note that a shorter proof of this theorem can be obtained by using only the algebraic structure that was mentioned following Theorem 2.1.

The last theorem corresponds to the Meobius inversion formula in the theory of the Dirichlet product.

From the familiar representation of Euler's function as a Dirichlet product, we are led to the following generalized ϕ function.

DEFINITION 2.2: $\phi_k^*(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d)\delta.$

By Theorem 2.4 and the definition of $\phi_k^*(n)$, we have immediately

THEOREM 2.5: $\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \phi_k^*(n) = n.$

Also, by Theorem 2.2, we have

REMARK 2.1: $\phi_k^*(n)$ is multiplicative.

We now define the k-prime analog of the square-free numbers. An integer n is said to be "k-square-free" provided that if a prime p divides n, then the multiplicity of p is in the range $\{1, 2, \ldots, k-1, k+1, k+2, \ldots, 2k-1\}$. So if $q_k^*(n)$ denotes the characteristic function of the set Q_k^* of k-square-free numbers, then $q_k^*(n)$ is multiplicative and, for prime powers p^e, has the evaluation

$$q_k^*(p^e) = \begin{cases} 1 & \text{if } e \in \{0, 1, \ldots, k-1, k+1, k+2, \ldots, 2k-1\} \\ 0 & \text{otherwise.} \end{cases}$$

3. FURTHER RESULTS

The algebraic results above coincide with classical results in the study of arithmetical functions. Another area of interest is in the area of analytic number theory. An important technique for obtaining estimates on the asymptotic average of an arithmetical function f is to express f as a Dirichlet product of functions g and h. Therefore, it is natural to investigate the possibility of expressing a function f as a product of two functions under our new convolution, and whenever such a representation exists, to use it to obtain asymptotic results for f. This would allow us to investigate certain functions which do not arise naturally as a Dirichlet product. Some results have been obtained by this method but more refinements are required.

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COMBINATORIAL ASPECTS OF AN INFINITE PATTERN OF INTEGERS

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1. INTRODUCTION

In two previous papers, [3] and [4], certain basic properties of the sequence $\{A_n(x)\}$ defined by

$$(1.1)\qquad \begin{aligned} &A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1, \text{ and} \\ &A_n(x) = xA_{n-2}(x) - A_{n-4}(x) \end{aligned}$$

were obtained by the authors.

Here, we wish to investigate further properties of this sequence using as our guide some of the numerical information given by L. G. Wilson [5]. Terminology and notation of [3] and [4] will be assumed to be available to the reader. In particular, let