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POLYGONAL PRODUCTS OF POLYGONAL NUMBERS AND THE PELL EQUATION

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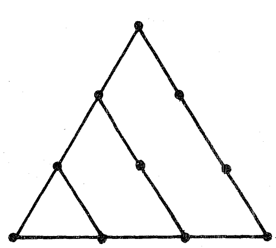
1. INTRODUCTION

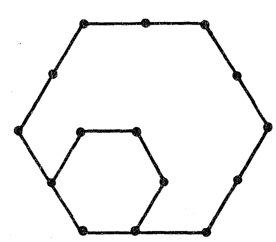
The kth polygonal number of order n (or the kth n-gonal number) P_k^n is given by the equation

$$P_k^n = P_k^n = k[(n - 2)(k - 1) + 2]/2.$$

Diophantus (c. 250 A.D.) noted that if the arithmetic progression with first term 1 and common difference $n - 2$ is considered, then the sum of the first k terms is P_k^n. The usual geometric realization, from which the name derives, is obtained by considering regular polygons with n sides sharing a common angle and having points at equal distances along each side with the total number of points being P_k^n. Two pictorial illustrations follow.



$P_4^3 = 10$



$P_3^6 = 15$

The first forty pages of Dickson's *History of Number Theory*, Vol. II, is devoted to results on polygonal numbers.

In 1968, W. Sierpiński [6] showed that there are infinitely many triangular numbers which at the same time can be written as the sum, the difference, and the product of other triangular numbers. It is easy to show that $4(m^2 + 1)^2$ is the sum, difference, and product of squares. Since then, several authors have proved similar results for sums and differences of other polygonal numbers. R. T. Hansen [2] considered pentagonal numbers, W. J. O'Donnell [4, 5] considered hexagonal and septagonal numbers, and S. Ando [1] proved that for any n infinitely many n-gonal numbers can be written as the sum and difference of other n-gonal numbers. Although Hansen gives several examples of pentagonal numbers written as the product of two other pentagonal numbers, the existence of an infinite class was left in doubt.

In this paper we show that for every n there are infinitely many n-gonal numbers that can be written as the product of two other n-gonal numbers, and in fact show how to generate infinitely many such products. We suspect that our method does not generate all of the solutions for every n, but we have not tried to prove this. Perhaps some reader will be challenged to try to find a product which is not generated by our method. Moreover, except for $n = 3$ and 4, it is still not known whether there are infinitely many n-gonal numbers which at the same time can be written as the sum, difference, and product of n-gonal numbers.

Our proof uses the well-known theory of the Pell equation. We also use a result (not found by us in the literature) on the existence of infinitely many solutions of a Pell equation satisfying a congruence condition, given that one solution exists satisfying the congruence condition. In Section 2 we note some facts about the Pell equation and prove this latter result. In Section 3 we prove our theorem on products of polygonal numbers.

2. THE PELL EQUATION

Although it was first issued by Fermat as a challenge problem, and a complete theory was given by Lagrange, the equation

$$u^2 - Dv^2 = M, \tag{1}$$

where D is not a perfect square, is usually called the Pell equation. The special case

$$u^2 - Dv^2 = 1 \tag{2}$$

always has an infinite number of solutions when D is not a square. In fact, if (u_1, v_1) is the least solution of (2), then any solution (u_j, v_j) is given (see, e.g. [3, pp. 139-48]) by the equation

$$u_j + \sqrt{D}v_j = (u_1 + \sqrt{D}v_1)^j. \tag{3}$$

Also, it is easy to see that if (u^*, v^*) is any particular solution of (1), then (u_j^*, v_j^*), given by

$$u_j^* + \sqrt{D}v_j^* = (u^* + \sqrt{D}v^*)(u_j + \sqrt{D}v_j), \tag{4}$$

is also a solution. Thus, we can generate infinitely many solutions to (1) if we can find one solution.

In what follows, Z^+ denotes the positive integers and $(a, b) \equiv (c, d) \pmod{m}$ means that $a \equiv c$ and $b \equiv d \pmod{m}$. We first prove a result which is heavily dependent upon the representability given by (3) of the solutions to (2).

THEOREM 1: If $D \in Z^+$ is not a square, then for any $m \in Z^+$ there are infinitely many integral solutions to the Pell equation

$$u^2 - Dv^2 = 1 \text{ with } (u, v) \equiv (1, 0) \pmod{m}.$$

PROOF: Suppose (u_1, v_1) is the least solution to (2) and (u_j, v_j) is the solution given by (3). Since there are only m^2 distinct ordered pairs of integers modulo m, there must be $j, \ell \in Z$ such that $(u_j, v_j) \equiv (u_\ell, v_\ell) \pmod{m}$. Using (3) we notice that, for any $t \in Z$,

$$u_t + \sqrt{D}v_t = (u_1 + \sqrt{D}v_1)(u_{t-1} + \sqrt{D}v_{t-1})$$

so

$$u_t = u_1 u_{t-1} + Dv_1 v_{t-1} \quad \text{and} \quad v_t = v_1 u_{t-1} + u_1 v_{t-1}.$$

Applying these equations to the above congruence, we deduce

$$u_1 u_{j-1} + Dv_1 v_{j-1} \equiv u_1 u_{\ell-1} + Dv_1 v_{\ell-1} \pmod{m} \tag{5}$$

and

$$v_1 u_{j-1} + u_1 v_{j-1} \equiv v_1 u_{\ell-1} + u_1 v_{\ell-1} \pmod{m}. \tag{6}$$

Multiplying (6) by u_1 and subtracting v_1 times (5), we have

$$(u_1^2 - Dv_1^2)v_{j-1} \equiv (u_1^2 - Dv_1^2)v_{\ell-1} \pmod{m},$$

or since $u_1^2 - Dv_1^2 = 1$,

$$v_{j-1} \equiv v_{\ell-1} \pmod{m}.$$

Similarly, u_1 times (5) minus Dv_1 times (6) yields

$$u_{j-1} \equiv v_{\ell-1} \pmod{m},$$

so in fact

$$(u_{j-1}, v_{j-1}) \equiv (u_{\ell-1}, v_{\ell-1}) \pmod{m}.$$

We can conclude, therefore, that for $K = |j - \ell|$,

$$(u_0, v_0) \equiv (u_{sK}, v_{sK}) \pmod{m}$$

for any $s \in Z^+$. But $u_0 = 1$ and $v_0 = 0$, so the theorem is proved.

As a corollary we can prove the following theorem about the general Pell equation showing infinitely many solutions in prescribed congruence classes.

THEOREM 2: If $m, D \in Z^+$, D is not a square, and the Pell equation $u^2 - Dv^2 = M$ has a solution

$$(u^*, v^*) \equiv (a, b) \pmod{m},$$

then it has infinitely many solutions

$$(u_t^*, v_t^*) \equiv (a, b) \pmod{m}.$$

PROOF: Let (u^*, v^*) be the solution to (1) provided in the hypothesis, and, for $t \in Z^+$, let (u_t, v_t) be solutions of (2) guaranteed by Theorem 1, that is,

$$(u_t, v_t) \equiv (1, 0) \pmod{m}.$$

Then the solutions (u_t^*, v_t^*) of (1) obtained from these solutions by applying (4) are such that

$$u_t^* = u^* u_t + Dv^* v_t \equiv a \cdot 1 + D \cdot b \cdot 0 \equiv a \pmod{m}$$

and

$$v_t^* = v^* u_t + u^* v_t \equiv b \cdot 1 + a \cdot 0 \equiv b \pmod{m},$$

as desired.

The following corollary follows by taking m in the previous theorem to be the least common multiple of m_1 and m_2.

COROLLARY: If $m_1, m_2, D \in Z^+$, D is not a square, and $a^2 - Db^2 = M$, then there are infinitely many solutions to the Pell equation $u^2 - Dv^2 = M$ with $u \equiv a \pmod{m_1}$ and $v \equiv b \pmod{m_2}$.

3. POLYGONAL PRODUCTS

In this section we first show that any nonsquare n-gonal number is infinitely often the quotient of two n-gonal numbers. The theorem that n-gonal products are infinitely often n-gonal and a remark on the solvability of a related equation complete this section.

THEOREM 3: If the n-gonal number $P = P_s$ is not a square, then there exist infinitely many distinct pairs (P_x, P_y) of n-gonal numbers such that

$$P_x = P_s P_y. \tag{7}$$

PROOF: Recalling that $P_x = \frac{1}{2}x[(n-2)(x-1)+2]$ and setting $n - 2 = r$, Eq. (7) becomes

$$rx^2 - (r-2)x = P[ry^2 - (r-2)y].$$

Multiplying by $4r$ to complete the square gives

$$(2rx - (r-2))^2 - (r-2)^2 = P[(2ry - (r-2))^2 - (r-2)^2].$$

Setting

$$u = 2rx - (r-2), \quad v = 2ry - (r-2), \tag{8}$$

we get the Pell equation

$$u^2 - Pv^2 = M, \tag{9}$$

with $M = (r-2)^2 - P(r-2)^2$.

Thus, in order to ensure infinitely many solutions (x, y) to (7), it suffices to have infinitely many solutions (u, v) to (9) for which the pair (x, y) obtained from (8) is integral. Put another way, it suffices to show the existence of infinitely many solutions (u^*, v^*) of (9) for which the congruence

$$(u^*, v^*) \equiv (-(r-2), -(r-2)) \equiv (r+2, r+2) \pmod{2r}$$

holds.

But notice that, since $P_1 = 1$, a particular solution of (7) is $x = s$, $y = 1$, and these values of x and y give

$$u = (2s-1)r + 2, \quad v = r + 2,$$

as a particular solution of (9). Thus, we have a solution (u^*, v^*) of (9) with $(u^*, v^*) \equiv (r+2, r+2) \pmod{2r}$. Theorem 2 guarantees the infinitely many solutions we are seeking.

Our final theorem is now a straightforward corollary.

THEOREM 4: For any $n \geq 3$, there are infinitely many n-gonal numbers which can be written as a product of two other n-gonal numbers.

PROOF: The case $n = 4$ is trivial. By the previous theorem, we need only show that P_s is not a square for some s. But for $n \neq 4$, at least one of $P_2 = n$ and $P_9 = 9(4n - 7)$ is not a square.

REMARK 1: We originally tried to prove that

$$P_k = k[(n-2)(k-1)+2]/2 = P_x \cdot P_y$$

infinitely often by setting $P_x = k$ and

$$P_y = ((n-2)(P_x - 1) + 2)/2,$$

and solving the Pell equation that results from this last equation. This method works if $n \neq 2t^2 + 2$, and thus, for these values of n, there are infinitely many solutions to the equation $P_{P_x} = P_x P_y$.

REMARK 2: There are 51 solutions of $P_x^3 = P_s^3 P_y^3$ with $P_x < 10^6$. There are 43 solutions of $P_x^n = P_s^n P_y^n$ with $5 \leq n \leq 36$ and $P_x^n < 10^6$. In just two of these, $x = P_s$:

$$P_{477}^5 = P_{18}^5 P_{22}^5 \quad \text{and} \quad P_{946}^6 = P_{22}^6 P_{31}^6.$$

For $36 \leq n \leq 720$, there are no solutions with $P_x^n < 10^6$.

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WAITING FOR THE *K*TH CONSECUTIVE SUCCESS AND THE FIBONACCI SEQUENCE OF ORDER *K*

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1. INTRODUCTION AND SUMMARY

In the sequel, k is a fixed integer greater than or equal to 2, and n is an integer as specified. Let N_k be a random variable denoting the number of trials until the occurrence of the kth consecutive success in independent trials with constant success probability p $(0 < p < 1)$. Shane [6] and Turner [7] considered the problem of obtaining the distribution of N_k. The first author found a formula for $P[N_k = n]$ $(n \geq k)$, as well as for $P[N_k \leq x]$ $(x \geq k)$, in terms of the polynacci polynomials of order k in p. Turner derived a formula for $P[N_k = n + k - 1]$ $(n \geq 1)$ in terms of the entries of the Pascal-T triangle. Both Shane and Turner first treated the special cases $p = 1/2$, $k = 2$, and $p = 1/2$, general k. For these cases, their formulas coincide.

Presently, we reconsider the problem and derive a new and simpler formula for $P[N_k = n + k]$ $(n \geq 0)$, in terms of the multinomial coefficients (see Theorem 3.1). The method of proof is also new. Interestingly enough, our formula includes as corollaries the special formulas of Shane and Turner. We present these results in Section 3. In Section 2, we obtain an expansion of the Fibonacci sequence of order k in terms of the multinomial coefficients (see Theorem 2.1), which is of interest in its own right and instrumental in deriving one of the corollaries.

2. THE FIBONACCI SEQUENCE OF ORDER K

In this section, we consider the Fibonacci sequence of order k and derive an expansion of it, in terms of the multinomial coefficients.