ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of

$$x^2 - x - 1 = 0.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-466 Proposed by Herta T. Freitag, Roanoke, VA

Let $A_n = 1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \cdots + (-1)^{n-1}n(n+1)$.

(a) Determine the values of n for which $2A_n$ is a perfect square.
(b) Determine the values of n for which $|A_n|/2$ is the product of two consecutive positive integers.

B-467 Proposed by Herta T. Freitag, Roanoke, VA

Let A_n be as in B-466 and let $B_n = \sum_{i=1}^{n} \sum_{k=1}^{i} k$. For which positive integers n is $|A_n|$ an integral divisor of B_n?

B-468 Proposed by Miha'ly Bencze, Brasov, Romania

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

B-469 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

Describe the appearance in base F_n notation of:

(a) $1/F_{n-1}$ for $n \geq 5$; (b) $1/F_{n+1}$ for $n \geq 3$.

B-470 Proposed by Larry Taylor, Rego Park, NY

Find positive integers a, b, c, r, and s and choose each of G_n, H_n, I_n to be F_n or L_n so that aG_n, bH_{n+r}, cI_{n+s} are in arithmetic progression for $n \geq 0$ and this progression is 6, 6, 6 for some n.

B-471 Proposed by Larry Taylor, Rego Park, NY

Do there exist positive integers d and t such that aG_n, bH_{n+r}, cI_{n+s}, dJ_{n+t} are in arithmetic progression, with J_n equal to F_n or L_n and everything else as in B-470?

SOLUTIONS

Lucas Analogue of Cosine Identity

B-442 Proposed by P. L. Mana, Albuquerque, NM

The identity $2\cos^2\theta = 1 + \cos(2\theta)$ leads to the identity

$$8\cos^4\theta = 3 + 4\cos(2\theta) + \cos(4\theta).$$

Are there corresponding identities on Lucas numbers?

Solution by Sahib Singh, Clarion State College, Clarion, PA

Yes; $L_{2n} = a^{2n} + b^{2n} = (a^n + b^n)^2 - 2(-1)^n = L_n^2 - 2(-1)^n$. Hence

(1) $$L_n^2 = L_{2n} + 2(-1)^n.$$

Using (1), $L_{2n}^2 = L_{4n} + 2$. Again using (1), the above equation reduces to

$$(L_n^2 - 2(-1)^n)^2 = L_{4n} + 2,$$

which yields

(2) $$L_n^4 = 6 + 4(-1)^n L_{2n} + L_{4n}.$$

Equations (1) and (2) are the required identities.

Also solved by Paul S. Bruckman, Paul F. Byrd, Herta T. Freitag, Calvin L. Gardner, Bob Prielipp, M. Wachtel, Gregory Wulczyn, and the proposer.

Lucas Products Identity

B-443 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

For all integers n and w with w odd, establish the following:

$$L_{n+2w}L_{n+w} - 2L_w L_{n+w}L_{n-w} - L_{n-w}L_{n-2w} = L_n^2(L_{3w} - 2L_w).$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given equation is equivalent to:

$$L_{n+2w}L_{n+w} - L_{n-w}L_{n-2w} - L_n^2 L_{3w} = 2L_w(L_{n+w}L_{n-w} - L_n^2).$$

Using the identity $L_{n+w}L_{n-w} - L_n^2 = 5(-1)^{n+w}F_w^2$, the above equation becomes:

(1) $$L_{n+2w}L_{n+w} - L_{n-w}L_{n-2w} - L_n^2 L_{3w} = 10(-1)^{n+w}L_w F_w^2.$$

Using $L_n = a^n + b^n$, the left side of (1) becomes

$$2(-1)^{n+w}(L_w + L_{3w}).$$

Thus (1) reduces to $L_w + L_{3w} = 5L_w F_w^2$. Since w is odd, by using $L_n = a^n + b^n$, the above equation is true and we are done.

Also solved by Paul S. Bruckman, Herta T Freitag, Calvin L. Gardner, Bob Prielipp, M. Wachtel, and the proposer.

Generating Palindromes

B-444 Proposed by Herta T. Freitag, Roanoke, VA

In base 10, the palindromes (i.e., numbers reading the same forward or backward) 12321 and 112232211 are converted into new palindromes using

$$99[10^3 + 9(12321)] = 11077011 \quad \text{and} \quad 99[10^5 + 9(112232211)] = 100008800001.$$

Generalize on these to obtain a method or methods for converting certain palindromes in a general base b to other palindromes in base b.

Solution by Paul S. Bruckman, Concord CA

Let $\mathcal{P}_b$ denote the set of palindromes in base b. We will prove the following theorem.

THEOREM: If $m \geq 1$, let $P \in \mathcal{P}_b$ be given by

$$P \equiv \sum_{k=0}^{m-1} (b^k + b^{2m-k})\theta_k + b^m\theta_m \equiv (\theta_0\theta_1\theta_2 \ldots \theta_{m-1}\theta_m\theta_{m-1} \ldots \theta_1\theta_0) \,. \tag{1}$$

Moreover, suppose the digits θ_k satisfy the following conditions:

$$1 \leq \theta_0 \leq \theta_1 \leq b-1, \text{ if } m = 1;\ 1 \leq \theta_0 \leq \theta_1 \leq \theta_0 + \theta_1 \leq \theta_2 \leq b-1, \text{ if } m \geq 2; \tag{2}$$

$$0 \leq \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3} \leq b-1, \text{ if } 3 \leq k \leq m; \tag{3}$$

$$0 \leq \theta_{m-2} \leq \theta_m \leq b-1, \text{ if } m \geq 3. \tag{4}$$

Let

$$Q \equiv (b^2 - 1)(b^{m+1} + (b-1)P). \tag{5}$$

Then $Q \in \mathcal{P}_b$.

PROOF: $Q = (b-1)(b^{m+2} + b^{m+1}) + (b^3 - b^2 - b + 1)P$

$$= (b-1)(b^{m+2} + b^{m+1}) + \sum_{k=0}^{m-1}(b^{k+3} + b^{2m+3-k})\theta_k - \sum_{k=0}^{m-1}(b^{k+2} + b^{2m+2-k})\theta_k$$

$$- \sum_{k=0}^{m-1}(b^{k+1} + b^{2m+1-k})\theta_k + \sum_{k=0}^{m-1}(b^k + b^{2m-k})\theta_k + (b^{m+3} - b^{m+2} - b^{m+1} + b^m)\theta_m$$

$$= (b-1)(b^{m+2} + b^{m+1}) + \sum_{k=3}^{m+2}(b^k + b^{2m+3-k})\theta_{k-3} - \sum_{k=2}^{m+1}(b^k + b^{2m+3-k})\theta_{k-2}$$

$$- \sum_{k=1}^{m}(b^k + b^{2m+3-k})\theta_{k-1} + \sum_{k=0}^{m-1}(b^k + b^{2m+3-k})\theta_k + (b^{m+3} - b^{m+2} - b^{m+1} + b^m)\theta_m.$$

After some manipulation, this last expression simplifies to the following:

$$Q = (b^0 + b^{2m+3})\theta_0 + (b^1 + b^{2m+2})(\theta_1 - \theta_0) + (b^2 + b^{2m+1})(\theta_2 - \theta_1 - \theta_0)$$

$$+ \sum_{k=3}^{m}(b^k + b^{2m+3-k})(\theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3}) + (b^{m+1} + b^{m+2})(b - 1 - \theta_m + \theta_{m-2}).$$

If we make the following definitions

$$c_0 \equiv \theta_0, \tag{6}$$

$$c_1 \equiv \theta_1 - \theta_0, \tag{7}$$

$$c_2 \equiv \theta_2 - \theta_1 - \theta_0, \tag{8}$$

$$c_k \equiv \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3},\ 3 \leq k \leq m, \tag{9}$$

and

(10) $$c_{m+1} \equiv b - 1 - \theta_m + \theta_{m-2},$$

we may express Q as follows:

(11) $$Q = \sum_{k=0}^{m+1} (b^k + b^{2m+3-k})c_k.$$

Moreover, we see from conditions (2), (3), and (4) that, for $0 \leq k \leq m + 1$, the inequalities $0 \leq c_k \leq b - 1$ (with $c_0 \geq 1$) are satisfied, i.e., the c_k's are digits in base b. Therefore, $Q = (c_0 c_1 c_2 \ldots c_m c_{m+1} c_{m+1} \ldots c_1 c_0)_b \in \mathcal{P}_b$. Q.E.D.

This result is readily specialized to decimal numbers by setting $b = 10$ in the theorem. Thus, if $P \in \mathcal{P}_{10}$ is given by (1)-(4), with $b = 10$, then

$$Q = 99(10^{m+1} + 9P)$$

is a palindrome in base 10.

Also solved by the proposer.

Simple Form

B-445 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, PA

Show that

$$5F_{2n+2}^2 + 2L_{2n}^2 + 5F_{2n-2}^2 = L_{2n+2}^2 + 10F_{2n}^2 + L_{2n-2}^2,$$

and find a simpler form for these equal expressions.

Solution by F. D. Parker, St. Lawrence University, Canton, NY

Using the identities $L_n = a^n + b^n$, $F_n = (a^n - b^n)/\sqrt{5}$, and $ab = -1$, both sides reduce to

$$a^{4n+4} + b^{4n+4} + 2a^{4n} + 2b^{4n} + a^{4n-4} + b^{4n-4},$$

which can be written

$$L_{4n+4} + 2L_{4n} + L_{4n-4}.$$

Using $L_n = L_{n-1} + L_{n-2}$, we see that this is equal to $9L_{4n}$.

Also solved by Paul S. Bruckman, Herta T. Freitag, Calvin L. Gardner, Graham Lord, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

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