average of an arithmetical function f is to express f as a Dirichlet product of functions g and h. Therefore, it is natural to investigate the possibility of expressing a function f as a product of two functions under our new convolution, and whenever such a representation exists, to use it to obtain asymptotic results for f. This would allow us to investigate certain functions which do not arise naturally as a Dirichlet product. Some results have been obtained by this method but more refinements are required.

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COMBINATORIAL ASPECTS OF AN INFINITE PATTERN OF INTEGERS

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1. INTRODUCTION

In two previous papers, [3] and [4], certain basic properties of the sequence $\{A_n(x)\}$ defined by

$$(1.1)\quad \begin{aligned} &A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1, \text{ and} \\ &A_n(x) = xA_{n-2}(x) - A_{n-4}(x) \end{aligned}$$

were obtained by the authors.

Here, we wish to investigate further properties of this sequence using as our guide some of the numerical information given by L. G. Wilson [5]. Terminology and notation of [3] and [4] will be assumed to be available to the reader. In particular, let

(1.2) $$\beta_i = 4\cos^2\frac{i\pi}{2n},$$
then
(1.3) $$\beta_{n+i} = \beta_{n-i},$$
(1.4) $$\beta_i - 2 = 2\cos\frac{i\pi}{n},$$
and
(1.5) $$(\beta_i - 2)^2 = \beta_{2i}.$$

The main result in this paper is Theorem 6. Besides the proof given, another proof is available.

2. PROPERTIES OF $A_k(\beta_i - 2)$

The following theorems generalize computational details in [5]. In Theorems 1 and 2, we use results in [3] and [4] with the Chebyshev polynomial of the second kind, $U_n(x)$.

THEOREM 1: $$A_{2n-1}(\beta_i - 2) = \begin{cases} +1 & (i \text{ odd}) \\ -1 & (i \text{ even}) \end{cases} \quad (i = 1, 2, 3, \ldots, n-1).$$

PROOF:
$$\begin{aligned} A_{2n-1}(\beta_i - 2) &= A_{2n}(\beta_i - 2) + A_{2n-2}(\beta_i - 2) \\ &= U_{n-1}\left(\cos\frac{i\pi}{n}\right) + U_{n-2}\left(\cos\frac{i\pi}{n}\right) \text{ by (1.4) and [4]} \\ &= \frac{\sin\left(n\cdot\frac{i\pi}{n}\right) + \sin(n-1)\frac{i\pi}{n}}{\sin\frac{i\pi}{n}} \\ &= \pm 1 \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}. \end{aligned}$$

E.g., $$A_9\ 2\cos\frac{\pi}{5} = 1.$$

THEOREM 2: $$A_r(\beta_i - 2) = \pm A_{2n-r}(\beta_i - 2) \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}$$
$$(r \text{ odd};\ i = 1, 2, 3, \ldots, n-1).$$

PROOF:
$$\begin{aligned} A_r(\beta_i - 2) &= A_{r+1}(\beta_i - 2) + A_{r-1}(\beta_i - 2) \\ &= \frac{U_{r-1}}{2}\left(\cos\frac{i\pi}{n}\right) + \frac{U_{r-3}}{2}\left(\cos\frac{i\pi}{n}\right) \text{ by (1.4) and [4]} \\ &= \frac{\sin\left(\frac{r+1}{2}\right)\frac{i\pi}{n} + \sin\left(\frac{r-1}{2}\right)\frac{i\pi}{n}}{\sin\frac{i\pi}{n}} = \frac{\sin\frac{ri\pi}{2n}}{\sin\frac{i\pi}{2n}} \\ &= \pm\frac{\sin\left(n - \frac{r}{2}\right)\frac{i\pi}{n}}{\sin\frac{i\pi}{2n}} \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases} \\ &= \pm\left[U_{n-\frac{r+1}{2}}\left(\cos\frac{i\pi}{n}\right) + U_{n-\frac{r+3}{2}}\left(\cos\frac{i\pi}{n}\right)\right] \\ &= \pm[A_{2n-r+1}(\beta_i - 2) + A_{2n-r-1}(\beta_i - 2)] \text{ by (1.4) and [4]} \\ &= \pm A_{2n-r}(\beta_i - 2) \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}. \end{aligned}$$

COROLLARY 1: When $i = 1$, $A_r\, 2\left(\cos \frac{\pi}{n}\right) = A_{2n-r}\left(2 \cos \frac{\pi}{n}\right)$.

E.g., $A_3\left(2 \cos \frac{\pi}{5}\right) = A_7\left(2 \cos \frac{\pi}{5}\right) = \dfrac{\sin \frac{3\pi}{10}}{\sin \frac{\pi}{10}} = 2 \cos \frac{\pi}{5} + 1 = \dfrac{3 + \sqrt{5}}{2} = \left(\dfrac{1 + \sqrt{5}}{2}\right)^2$.

COROLLARY 2: $A_1(\beta_i - 2), A_3(\beta_i - 2), \ldots, A_{2n-1}(\beta_i - 2)$ for a cycle of period n.

E.g., for $n = 6$, $i = 1$, $A_1 = A_{11} = 1$, $A_3 = A_9 = 1 + \sqrt{3}$, $A_5 = A_7 = 2 + \sqrt{3}$.

Our next theorem involves $\phi(n)$, Euler's ϕ-function.

THEOREM 3: Let n be odd and $m = \frac{1}{2}\phi(n)$, then $\beta_{2^m} - 2 = -(\beta_1 - 2)$.

PROOF: By the Fermat-Euler Theorem, since $(2, n) = 1$, it follows that

$$2^m \equiv \pm 1 \pmod{n}.$$

Hence, there exists an odd integer t such that $2^m = nt \pm 1$. Therefore,

$$\begin{aligned}\beta_{2^m} - 2 = 2 \cos 2^m\left(\frac{\pi}{n}\right) &= 2 \cos(nt \pm 1)\frac{\pi}{n} \\ &= 2 \cos \pi t \cos\left(\pm \frac{\pi}{n}\right) \\ &= -(\beta_1 - 2).\end{aligned}$$

COROLLARY 3: When n is even, just one operator ("square and subtract 2") produces the $\beta_1 - 2$ for $n/2$.

This is obvious, because $(\beta_1 - 2)^2 - 2 = 2 \cos \frac{2\pi}{n} = 2 \cos \frac{\pi}{n/2}$.

3. SEMI-INFINITE NUMBER PATTERNS

Consider the pattern of numbers and their mode of generation given in Table 1 for a fixed number $k = 5$ of columns (Wilson [5]).

Column m / Row n	1		2		3		4		5
0	1		1		1		1		1
		2		2		2		2	
1	2		4		4		4		2
		6		8		8		6	
2	6		14		16		14		6
		20		30		30		20	
3	20		50		60		50		20
		70		110		110		70	
4	70		180		220		180		70
		250		400		400		250	
5	250		650		800		650		250
		900		1450		1450		900	
6	900		2350		2900		2350		900
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

Table 1. Pattern of Integers for $k = 5$

Designate the row number by n and the column number by m ($n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots, k$). The element in row n and column m is denoted by U_{nm}.

From Table 1, the following information may be gleaned:

$$\begin{bmatrix} U_{n1} \\ U_{n2} \\ U_{n3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} U_{n-1,1} \\ U_{n-1,2} \\ U_{n-1,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.1}$$

$$U_{nm} = 5(U_{n-1,m} - U_{n-2,m}),\ n > 2 \text{ and} \tag{3.2}$$

$$\begin{cases} U_{n1} = U_{n5} = \dfrac{2}{\sqrt{5}}(\alpha a^{n-1} - \beta b^{n-1}) \\ U_{n2} = U_{n4} = \dfrac{2}{\sqrt{5}}(Aa^{n-1} - Bb^{n-1}), \quad n \geq 1, \\ \qquad\quad U_{n3} = \dfrac{2}{\sqrt{5}}(Ca^{n-1} - Db^{n-1}) \end{cases} \tag{3.3}$$

where

$$\begin{cases} a = \frac{1}{2}(5 + \sqrt{5}),\ b = \frac{1}{2}(5 - \sqrt{5}) \\ \alpha = \frac{1}{2}(1 + \sqrt{5}),\ \beta = \frac{1}{2}(1 - \sqrt{5}) \\ A = 2 + \sqrt{5}, \quad B = 2 - \sqrt{5} \\ C = 3 + \sqrt{5}, \quad D = 3 - \sqrt{5} \end{cases} \tag{3.4}$$

so that $A = 2\alpha + 1$, $B = 2\beta + 1$, $C = 2(\alpha + 1) = A + 1$, $D = 2(\beta + 1) = B + 1$.

It follows from (3.3) and (3.4) that

$$\lim_{n\to\infty}\left(\frac{U_{n2}}{U_{n1}}\right) = \frac{A}{\alpha} = \alpha + 1 = A_3\left(2\cos\frac{\pi}{5}\right), \tag{3.5}$$

and

$$\lim_{n\to\infty}\left(\frac{U_{n3}}{U_{n1}}\right) = \frac{C}{\alpha} = 2\alpha = A_5\left(2\cos\frac{\pi}{5}\right). \tag{3.6}$$

Extending Table 1 to the case $k = 6$, so that now, for example, $U_{51} = 252$ and $U_{43} = 236$, we eventually derive $U_{nm} = 6U_{n-1,m} - 9U_{n-2,m} + 2U_{n-3,m}$; thus

$$\begin{cases} U_{n1} = \frac{1}{3}\{2^n + (2 + \sqrt{3})^n + (2 - \sqrt{3})^n\} = U_{n6} \\ U_{n2} = \frac{1}{3}\{2^n + (1 + \sqrt{3})(2 + \sqrt{3})^n + (1 - \sqrt{3})(2 - \sqrt{3})^n\} = U_{n5} \\ U_{n3} = \frac{1}{3}\{-1.2^n + (2 + \sqrt{3})(2 + \sqrt{3})^n + (2 - \sqrt{3})(2 - \sqrt{3})^n\} = U_{n4}, \end{cases} \tag{3.7}$$

whence

$$\lim_{n\to\infty}\left(\frac{U_{n2}}{U_{n1}}\right) = \sqrt{3} + 1 = A_3\left(2\cos\frac{\pi}{6}\right), \tag{3.8}$$

and

$$\lim_{n\to\infty}\left(\frac{U_{n3}}{U_{n1}}\right) = 2 + \sqrt{3} = A_5\left(2\cos\frac{\pi}{6}\right). \tag{3.9}$$

Results (3.5), (3.6), (3.8), and (3.9) suggest a connection between various limits of ratios (as $n \to \infty$) and corresponding $A_r\left(2\cos\frac{\pi}{k}\right)$. This link is developed in the next section. [In passing, we remark that for $k = 9$, $n = 13$, we calculate to two decimal places that

$$\frac{U_{13,2}}{U_{13,1}} = 2.85,\ A_3\left(2\cos\frac{\pi}{9}\right) = 2.88,$$

the common value to which they aspire as $n \to \infty$, $k \to \infty$ being 3 (cf. Theorem 6).]

4. AN INFINITE NUMBER PATTERN

For $1 \le m \le k$, we find [cf. (3.1)]

$$(4.1)\qquad \begin{cases} U_{nm} = U_{n-1,m-1} + 2U_{n-1,m} + U_{n-1,m+1} & 1 < m < k \\ U_{n1} = U_{n-1,1} + U_{n-1,2} & m = 1 \\ U_{nk} = U_{n-1,k} + U_{n-1,k-1} & m = k \end{cases}$$

with $U_n \ = U_{n,k+1-m}$. Also

$$(4.2)\qquad \begin{cases} U_{nm} = \displaystyle\sum_{r=1}^{[k/2]} (-1)^{r-1} v_{kr} U_{n-r,m} & n > [k/2] \\ U_{n1} = \dbinom{2n}{n} & n \le k-1 \end{cases}$$

in which v_{nm} is an element of an array in row n and column m defined by

$$(4.3)\qquad \begin{aligned} v_{nm} &= v_{n-1,m} + v_{n-2,m-1} && n \ge 2m \\ v_{nm} &= 0 && n < 2m \\ v_{n1} &= n,\ v_{n0} = 1,\ v_{0m} = 0,\ v_{2n,2n-1} = 2. \end{aligned}$$

For example, if

$$k = 6,\ U_{nm} = 6U_{n-1,m} - 9U_{n-2,m} + 2U_{n-3,m},$$

and if

$$k = 9,\ U_{nm} = 9U_{n-1,m} - 27U_{n-2,m} + 30U_{n-3,m} - 9U_{n-4,m}.$$

We look briefly at the $\{v_{nk}\}$ in Section 5.

Notice in (4.2) that for $n \to \infty$, i.e., $k \to \infty$, U_{n1} are the *central binomial coefficients*.

Now let $n \to \infty$ and $k \to \infty$. We wish to obtain the limit of $U_{nm}/U_{nm'}$. But first, by easy calculation using (4.1) we derive

$$(4.4)\qquad \lim_{n\to\infty}\left(\frac{U_{n-1,1}}{U_{n1}}\right) = \frac{1}{4}.$$

THEOREM 4: $\displaystyle\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{n1}}\right) = 2m - 1.$

PROOF: The result is trivially true for $m = 1$. Assume the theorem is true for $m = p$. That is, assume

$$\lim_{n\to\infty}\left(\frac{U_{np}}{U_{n1}}\right) = 2p - 1.$$

We test this hypothesis for $m = p + 1$, using (4.1) several times. Now

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left(\frac{U_{n,\,p+1}}{U_{n1}}\right)\\
&= \lim_{n\to\infty}\left\{\frac{U_{n-1,\,p} + 2U_{n-1,\,p+1} + (U_{n,\,p+1} - U_{n-1,\,p} - 2U_{n-1,\,p+1})}{U_{n1}}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{2(U_{np} - U_{n-1,\,p-1} - 2U_{n-1,\,p}) + (U_{n,\,p+1} - 2U_{n-1,\,p+1})}{U_{n1}}\right\}\\
&= \lim_{n\to\infty}\left\{2\left(\frac{U_{np}}{U_{n1}} - \frac{U_{n-1,\,p-1}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}} - 2\frac{U_{n-1,\,p}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}}\right)\right.\\
&\qquad\left.+\left(\frac{U_{n,\,p+1}}{U_{n1}} - 2\frac{U_{n-1,\,p+1}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}}\right)\right\}\\
&= 2\left(2p - 1 - \left(\frac{2p-3}{4}\right) - 2\left(\frac{2p-1}{4}\right)\right) + \left(R - 2\frac{R}{4}\right) \quad \text{by (4.4) and the inductive hypothesis,}
\end{aligned}
$$

whence $R = 2p + 1$, which establishes the theorem.

COROLLARY 4: $\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{nm'}}\right) = \frac{2m-1}{2m'-1}$

THEOREM 5: $A_{2m-1}(\beta_1 - 2) = 2m - 1 = A_{2k-(2m-1)}(\beta_1 - 2),\ 1 \le m \le k,\ k \to \infty.$

PROOF:
$$
\begin{aligned}
A_{2m-1}(\beta_1 - 2) &= \frac{\sin(2m-1)\dfrac{i\pi}{2k}}{\sin\dfrac{i\pi}{2k}} = A_{2k-(2m-1)}(\beta_1 - 2) \text{ by Theorem 2}\\
&= 2m - 1
\end{aligned}
$$

on using a trigonometrical expansion for the numerator, simplifying, and then letting $k \to \infty$.

Clearly there is a connection between Theorems 4 and 5. We therefore assert:

THEOREM 6: $\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{n1}}\right) = A_{2m-1}(\beta_1 - 2) = 2m - 1 \qquad (k \to \infty).$

Observe that, with the aid of (4.1) and the manipulative technique of Theorem 4, we may deduce

$$
\lim_{n\to\infty}\left(\frac{U_{n-1,\,2}}{U_{n1}}\right) = \frac{3}{4},\ \lim_{n\to\infty}\left(\frac{U_{n-1,\,3}}{U_{n1}}\right) = \frac{5}{4},\ \lim_{n\to\infty}\left(\frac{U_{n-1,\,4}}{U_{n1}}\right) = \frac{7}{4},\ \ldots, \tag{4.5}
$$

and

$$
\lim_{n\to\infty}\left(\frac{U_{n-2,\,2}}{U_{n1}}\right) = \frac{3}{16},\ \lim_{n\to\infty}\left(\frac{U_{n-2,\,3}}{U_{n1}}\right) = \frac{5}{16},\ \lim_{n\to\infty}\left(\frac{U_{n-2,\,4}}{U_{n1}}\right) = \frac{7}{16},\ \ldots. \tag{4.6}
$$

Ultimately,

$$
\lim_{n\to\infty}\left(\frac{U_{n-r,\,m}}{U_{n1}}\right) = \frac{2m-1}{4^r}, \tag{4.7}
$$

from which Theorem 4 follows if we put $r = 0$.

This concludes the theoretical basis, with extensions, for the detailed numerical information given by Wilson [5].

5. SOME PROPERTIES OF $\{v_{kr}\}$

Define

$$\Delta v_{kr} = v_{kr} - v_{k-1,r}. \tag{5.1}$$

THEOREM 7: $\Delta^r v_{kr} = 1$.

PROOF: Use induction. When $r = 1$,

$$\Delta v_{k1} = k - (k - 1) = 1 \quad \text{by (5.1) and (4.3).}$$

Assume the result is true for $r = 2, 3, \ldots, s - 1$. Then

$$\begin{aligned} \Delta^s v_{ks} &= \Delta^{s-1}(\Delta v_{ks}) \\ &= \Delta^{s-1} \cdot (v_{k-1,s} + v_{k-2,s-1} - v_{k-1,s}) \quad \text{by (5.1) and (4.3)} \\ &= \Delta^{s-1} v_{k-2,s-1} \\ &= 1 \quad \text{from the inductive hypothesis.} \end{aligned}$$

Hence, the theorem is proved.

It can also be shown that

$$v_{nm} = \frac{n}{n-m}\binom{n-m}{m} = \binom{n-m}{m} + \binom{n-m-1}{m-1}, \tag{5.2}$$

whence

$$L_n = \sum_{m=0}^{[n/2]} v_{nm}, \tag{5.3}$$

in which L is the nth Lucas number defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \quad (n > 2)$$

with initial conditions $L_1 = 1$, $L_2 = 3$.

Another property is

$$\sum_{m=0}^{n} v_{n+m,m} = 3 \cdot 2^{n-1}. \tag{5.4}$$

Table 2 shows the first few values of v_{kr} (see Hoggatt & Bicknell [2], where the v_{kr} occur as coefficients in a list of Lucas polynomials).

k \ r	0	1	2	3	4	5
1	1					
2	1	2				
3	1	3				
4	1	4	2			
5	1	5	5			
6	1	6	9	2		
7	1	7	14	7		
8	1	8	20	16	2	
9	1	9	27	30	9	
10	1	10	35	50	25	2
11	1	11	44	77	55	11
	. . .	. . .	. . .	. . .	. . .	. . .

Table 2. Values of v_{kr} ($k = 1, 2, \ldots, 11$)

Coefficients in the generating difference equations (4.2), as k varies, appear in Table 2 if we alternate + and - signs. Corresponding characteristic polynomials occur in [4] as proper divisors, or as products of proper divisors. Refer to Hancock [1], also.

Further, it might be noted that, if we employ the recurrence relation in (4.1) repeated.y, we may expand U_{nm} binomially as

$$U_{nm} = U_{n-t,\, m-t} + \binom{2t}{1}U_{n-t,\, m-t+1} + \binom{2t}{2}U_{n-t,\, m-t+2} + \cdots$$
$$+ \binom{2t}{1}U_{n-t,\, m+t+1} + U_{n-t,\, m+t} \quad (1 \le t < n,\ 1 \le t < m).$$

This is because the original recurrence relation (4.1) for U_{nm} is "binomial" (t = 1), i.e., the coefficients are 1, 2, 1.

Finally, we remark that the row elements in the first column, U_{n1}, given in (4.2), are related to the *Catalan numbers* C_n by

$$U_{n1} = (n + 1)C_n. \tag{5.5}$$

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ONE-PILE TIME AND SIZE DEPENDENT TAKE-AWAY GAMES

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1. INTRODUCTION

In a one-pile take-away game, two players alternately remove chips from a single pile of chips. Depending on the particular formulation of play, a *constraint function* specifies the number of chips which may be taken from the pile in each position. The game ends when no move is possible. In *normal* (*misère*) play, the player who makes the final move wins (loses). Necessarily, one of the players has a strategy which can force a win.

In this *Quarterly*, Whinihan [7], Schwenk [5], and Epp & Ferguson [2] have analyzed certain one-pile take-away games which can be represented by an ordered