If γ is a class of increasing sequences of natural numbers (e.g., all increasing sequences or the arithmetic progressions), then we define

$$\Delta(\gamma) = \sup_{\omega \in \gamma} \delta(\omega).$$

Trivially, we obtain $\Delta(\gamma) \leq 2$.

The problem is to give better estimations for $\Delta(\gamma)$ in the general case or in the case where γ is the class of all arithmetic progressions.

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SOME IDENTITIES AND DIVISIBILITY PROPERTIES OF LINEAR SECOND-ORDER RECURSION SEQUENCES

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INTRODUCTION

Following Lucas [5], let P and Q be integers such that

$$\text{(i)} \qquad (P, Q) - 1 \quad \text{and} \quad D = P^2 + 4Q \neq 0.$$

Let the roots of

$$\text{(ii)} \qquad x^2 = Px + Q$$

be

$$\text{(iii)} \qquad a = (P + D^{\frac{1}{2}})/2, \; b = (P - D^{\frac{1}{2}})/2.$$

Consider the sequences

$$\text{(iv)} \qquad u^n = (a^n - b^n)/(a - b), \; v_n = a^n + b^n.$$

In this article, we examine sums of the form

$$\sum \binom{k}{j} x_n^j (Qx_{n-1})^{k-j} u_j,$$

where $x_n = u_n$ or v_n, and prove that

$$\text{g.c.d. } (u_n, u_{kn}/u_n) \text{ divides } k,$$

and that

$$\text{g.c.d. } (v_n, v_{kn}/v_n) \text{ divides } k \text{ if } k \text{ is odd.}$$

PRELIMINARIES

(1) $(u_n, Q) = (v_n, Q) = 1$
(2) $(u_n, u_{n-1}) = 1$
(3) $D = (a - b)^2$
(4) $P = a + b, \; Q = -ab$
(5) $v_n = u_{n+1} + Qu_{n-1}$
(6) $au_n + Qu_{n-1} = a^n, \; bu_n + Qu_{n-1} = b^n$
(7) $av_n + Qv_{n-1} = a^n(a - b), \; bv_n + Qv_{n-1} = -b^n(a - b)$
(8) $v_n = Pv_{n-1} + Qv_{n-2}$
(9) P even implies v_n even

(10) k odd implies $$v_{kn}/v_n = \sum_{j=0}^{(k-3)/2} v_{(k-1-2j)n}Q^{jn} + Q^{(k-1)n/2}$$

REMARKS: (1) is Carmichael [2, Th. I], and (2) follows from [2, Corollary to Th. VI]. (3) follows from (iii), (4) follows from (i) and (iii). (5) follows from (iv) and (4). (6) can be proved by induction, while (7) follows from (5) and (6). (8) follows from (iii) and (iv), (9) follows from Carmichael [2, Th. III], and (10) is Lucas [5, Eq. (44), p. 199].

THE MAIN THEOREMS

THEOREM 1: $u_{kn} = \sum_{j=1}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} u_j.$

PROOF: (iv) implies

$$(a - b)u_{kn} = a^{kn} - b^{kn} = (a^n)^k - (b^n)^k;$$

(6) implies

$$\begin{aligned}(a - b)u_{kn} &= (au_n + Qu_{n-1})^k - (bu_n + Qu_{n-1})^k\\ &= \sum_{j=0}^{k}\binom{k}{j}(au_n)^j(Qu_{n-1})^{k-j} - \sum_{j=0}^{k}\binom{k}{j}(bu_n)^j(Qu_{n-1})^{k-j}\\ &= \sum_{j=0}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}a^j - \sum_{j=0}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}b^j\\ &= \sum_{j=0}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}(a^j - b^j) = \sum_{j=1}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}(a^j - b^j).\end{aligned}$$

Therefore,

$$u_{kn} = \sum_{j=1}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}(a^j - b^j)/(a - b) = \sum_{j=1}^{k}\binom{k}{j}u_n^j(Qu_{n-1})^{k-j}u_j.$$

THEOREM 2: $(u_n, u_{kn}/u_n)\,|\,k.$

PROOF: Theorem 1 implies

$$u_{kn}/u_n = \sum_{j=1}^{k}\binom{k}{j}u_n^{j-1}(Qu_{n-1})^{k-j}u_j = k(Qu_{n-1})^{k-1} + \sum_{j=1}^{k}\binom{k}{j}u_n^{j-1}(Qu_{n-1})^{k-j}u_j.$$

Let $d = (u_n, u_{kn}/u_n)$, so that $d|u_n$, $d|u_{kn}/u_n$. Therefore, we have $d|k(Qu_{n-1})^{k-1}$. Now (1), (2) imply $(d, Q) = (d, u_{n-1}) = 1$. Therefore, $d|k$.

THEOREM 3: If k is odd, then

$$D^{(k-1)/2}v_{kn} = \sum_{j=1}^{k}\binom{k}{j}v_n^j(Qv_{n-1})^{k-j}u_j.$$

PROOF: Together, (iv) and (3) imply

$$\begin{aligned}(a - b)D^{(k-1)/2}v_{kn} &= (a - b)^k(a^{kn} + b^{kn})\\ &= (a - b)^k a^{kn} + (a - b)^k b^{kn}\\ &= \{(a - b)a^n\}^k - \{-(a - b)b^n\}^k.\end{aligned}$$

(7) implies

$$(a - b)D^{(k-1)/2}v_{kn} = (av_n + Qv_{n-1})^k - (bv_n + Qv_{n-1})^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} (av_n)^j (Qv_{n-1})^{k-j} - \sum_{j=0}^{k} \binom{k}{j} (bv_n)^j (Qv_{n-1})^{k-j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} a^j - \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} b^j$$

$$= \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j) = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j).$$

Therefore,

$$D^{(k-1)/2} v_{kn} = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j)/(a - b) = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} u_j.$$

LEMMA 1: $(v_n, v_{n-1}) = \begin{cases} 1 \text{ if } P \text{ is odd} \\ 2 \text{ if } P \text{ is even.} \end{cases}$

PROOF: Let $d = (v_n, v_{n-1})$, $d^* = (v_{n-1}, v_{n-2})$. (8) and (1) imply $d | d^*$, while (8) implies $d^* | d$, so that $d = d^*$. Repeating this argument $n - 1$ times, one has $d = (v_1, v_0)$. But (iv) and (4) imply $v_1 = P$ and $v_0 = 2$, so that $d = (P, 2)$. Therefore, P odd implies $d = 1$, P even implies $d = 2$.

LEMMA 2: k odd, P even imply v_{kn}/v_n odd.

PROOF: The hypothesis and (10) imply

$$v_{kn}/v_n - Q^{(k-1)n/2} = \sum_{j=0}^{(k-3)/2} v_{(k-1-2j)n} Q^{jn}$$

The hypothesis and (9) imply $v_{kn}/v_n - Q^{(k-1)n/2}$ is even, whereas the hypothesis and (9) imply Q is odd. Therefore, v_{kn}/v_n is odd.

LEMMA 3: $(v_{n-1}, v_n, v_{kn}/v_n) = 1$ if k is odd.

PROOF: Let $d = (v_n, v_{kn}/v_n)$, so that $d | v_n$ and (d, v_{n-1}) (v_n, v_{n-1}). Now Lemma 1 implies $(v_n, v_{n-1}) | 2$. Therefore $(d, v_{n-1}) | 2$. If P is even, Lemma 2 implies d is odd, which implies (d, v_{n-1}) is odd. Therefore $(d, v_{n-1}) = 1$. If P is odd, then Lemma 1 implies $(v_n, v_{n-1}) = 1$. Therefore $(d, v_{n-1}) = 1$.

THEOREM 4: k odd implies $(v_n, v_{kn}/v_n) | k$.

PROOF: The hypothesis and Theorem 2 imply

$$D^{(k-1)/2} v_{kn}/v_n = \sum_{j=1}^{k} \binom{k}{j} v_n^{j-1} (Qv_{n-1})^{k-j} u_j$$

$$= k(Qv_{n-1})^{k-1} + \sum_{j=2}^{k} \binom{k}{j} v_n^{j-1} (Qv_{n-1})^{k-j} u_j.$$

If $d = (v_n, v_{kn}/v_n)$, we have $d | k(Qv_{n-1})^{k-1}$. Now (1) and Lemma 3 imply

$$(d, Q) = (d, v_{n-1}) = 1.$$

Therefore, $d | k$.

CONCLUDING REMARKS

Theorem 1 generalizes a result pertaining to Fibonacci numbers, i.e., the case $P = Q = 1$, by Carlitz and Ferns [1, Eq. (1.6), p. 62] with $k = 0$; by Vinson [6, p. 38] with $r = 0$; and by Halton [3, Eq. (35), p. 35]. Theorem 2 generalizes Halton [4, Lem. XVI] as well as Carmichael [2, Th. XVII]. Theorems 1 and 3 remain valid if u_{kn}, v_{kn} are replaced by u_{kn+r}, v_{kn+r}, while u_j is replaced by u_{j+r}.

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POLYGONAL PRODUCTS OF POLYGONAL NUMBERS AND THE PELL EQUATION

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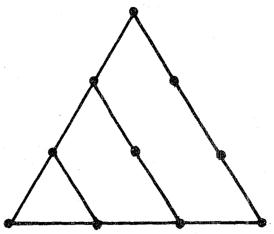
1. INTRODUCTION

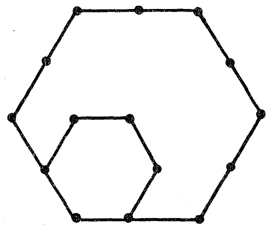
The kth polygonal number of order n (or the kth n-gonal number) P_k^n is given by the equation

$$P_k^n = P_k^n = k[(n - 2)(k - 1) + 2]/2.$$

Diophantus (c. 250 A.D.) noted that if the arithmetic progression with first term 1 and common difference $n - 2$ is considered, then the sum of the first k terms is P_k^n. The usual geometric realization, from which the name derives, is obtained by considering regular polygons with n sides sharing a common angle and having points at equal distances along each side with the total number of points being P_k^n. Two pictorial illustrations follow.



$P_4^3 = 10$



$P_3^6 = 15$

The first forty pages of Dickson's *History of Number Theory*, Vol. II, is devoted to results on polygonal numbers.