ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-472 *Proposed by Gerald E. Bergum, S. Dakota State University, Brookings, SD*

Find a sequence $\{T_n\}$ satisfying a second-order linear homogeneous recurrence $T_n = aT_{n-1} + bT_{n-2}$ such that every even perfect number is a term in $\{T_n\}$.

B-473 *Proposed by Philip L. Mana, Albuquerque, NM*

Let $a = L_{1000}$, $b = L_{1001}$, $c = L_{1002}$, $d = L_{1003}$. Is $1 + x + x^2 + x^3 + x^4$ a factor of $1 + x^a + x^b + x^c + x^d$? Explain.

B-474 *Proposed by Philip L. Mana, Albuquerque, NM*

Are there an infinite number of positive integers n such that

$$L_n + 1 \equiv 0 \pmod{2n}?$$

Explain.

B-475 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k$. Prove that $|S_3(n) - S_1^2(n)|$ is $2[(n + 1)/2]$ times a triangular number. Here [ ] denotes the greatest integer function.

B-476 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_k(n) = \sum_{j=1}^{n} (-1)^{j+1} j^k$. Prove that $|S_4(n) + S_2(n)|$ is twice the square of a triangular number.

B-477 *Proposed by Paul S. Bruckman, Sacramento, CA*

Prove that $\sum_{n=2}^{\infty} \text{Arctan}\, \frac{(-1)^n}{F_{2n}} = \frac{1}{2} \text{Arctan}\, \frac{1}{2}$.

SOLUTIONS

Casting Out 27's

B-446 *Proposed by Jerry M. Metzger, University of N. Dakota, Grand Forks, ND*

It is familiar that a positive integer n is divisible by 3 if and only if the sum of its digits is divisible by 3. The same is true for 9. For 27, this is false since, for example, 27 divides 1 + 8 + 9 + 9 but does not divide 1899. However, $27 | 1998$.

Prove that 27 divides the sum of the digits of n if and only if 27 divides one of the integers formed by permuting the digits of n.

Solution by Paul S. Bruckman, Concord, CA

Given

$$N = \sum_{k=0}^{m} a_k 10^k, \tag{1}$$

where the a_k's are decimal digits, let the sum of the digits be given by

$$s(N) = \sum_{k=0}^{m} a_k. \tag{2}$$

We begin by observing that the statement of the problem is false. The correct statement should read as follows: If 27 divides the sum of the digits of N, then 27 divides one of the integers formed by permuting the digits of N. The converse is clearly false, since, e.g., $27 | 27$ but $27 \nmid s(27) = 9$.

Suppose

$$27 | s(N). \tag{3}$$

The smallest positive integer N satisfying (3) is 999. Since $27 | 999$, we see that the (modified) proposition is verified for $N = 999$. We may therefore suppose $m \geq 3$.

Since $9 | s(N)$, thus $9 | N$. Let $\mathcal{P}_N$ denote the set of all possible integers M formed by permuting the digits of N. Since $s(M) = s(N)$ for all $M \in \mathcal{P}_N$, we see that $9 | M$ for all $M \in \mathcal{P}_N$. We will assume that $M \equiv \pm 9 \pmod{27}$ for all $M \in \mathcal{P}_N$ and show that this leads to a contradiction.

Given k $(0 \leq k \leq m-2)$, form $N_k^{(1)} \in \mathcal{P}_N$ and $N_k^{(2)} \in \mathcal{P}_N$ by merely permuting the triple (a_k, a_{k+1}, a_{k+2}) to (a_{k+1}, a_{k+2}, a_k) and (a_{k+2}, a_k, a_{k+1}), respectively. Then

$$\begin{aligned} N_k^{(1)} - N &= 10^k(a_{k+1} - a_k) + 10^{k+1}(a_{k+2} - a_{k+1}) + 10^{k+2}(a_k - a_{k+2}) \\ &\equiv 10^k\{a_{k+1} - a_k + 10(a_{k+2} - a_{k+1}) + 19(a_k - a_{k+2})\} \pmod{27} \\ &\equiv -9 \cdot 10^k(a_k + a_{k+1} + a_{k+2}) \pmod{27} \\ &\equiv -9(a_k + a_{k+1} + a_{k+2}) \pmod{27}. \end{aligned}$$

Similarly, we find that

$$N_k^{(2)} - N \equiv 9(a_k + a_{k+1} + a_{k+2}) \pmod{27}.$$

Having assumed that $N \equiv \pm 9 \pmod{27}$, we cannot have

$$a_k + a_{k+1} + a_{k+2} \equiv \pm 1 \pmod{3},$$

for we would then have either

$$N_k^{(1)} \equiv 0 \quad \text{or} \quad N_k^{(2)} \equiv 0 \pmod{27},$$

contradicting the assumption. Since k is arbitrary, we must therefore have

(4) $$a_k + a_{k+1} + a_{k+2} \equiv 0 \pmod{3},\ k = 0, 1, \ldots, m - 2.$$

Thus the sum of any three consecutive digits of N must be divisible by 3. But we see by symmetry that this same property must hold for all $M \in \mathcal{P}_N$. This can only be true if all the a_k's are congruent (mod 3).

Suppose, therefore, that

(5) $$a_k = 3b_k + r, \text{ where } b_k = 0, 1, 2, \text{ or } 3,\quad r = 0, 1, \text{ or } 2,\quad \text{with } b_k = 3 \text{ only if } r = 0.$$

Let

(6) $$B = \sum_{k=0}^{m} b_k 10^k.$$

Then $N = \sum_{k=0}^{m}(3b_k + r)10^k = 3B + r\sum_{k=0}^{m}10^k$, or

(7) $$N = 3B + \frac{r}{9}(10^{m+1} - 1).$$

Also,

$$s(N) = \sum_{k=0}^{m}(3b_k + r), \text{ or}$$

(8) $$s(N) = 3s(B) + r(m + 1).$$

We consider two (*a priori*) possibilities:

<u>(a) $m \equiv 0$ or $1 \pmod{3}$.</u> Since $3 | s(N)$, we see from (8) that $r = 0$. Hence, $N = 3B$ and $s(N) = 3s(B)$. But

$$27 | s(N) \Longrightarrow 9 | s(B) \Longrightarrow 9 | B \Longrightarrow 27 | N,$$

which contradicts our assumption. This leaves the only remaining possibility:

<u>(b) $m \equiv 2 \pmod 3$</u>. Let $m = 3t - 1$, $t \geq 2$. Note that

$$\frac{r}{9}(10^{m+1} - 1) = \frac{r}{9}(10^{3t} - 1) = r\sum_{k=0}^{3t-1} 10^k = 111r\sum_{k=0}^{t-1} 10^{3k}$$

$$\equiv 3r\sum_{k=0}^{t-1} 1 \pmod{27} \equiv 3rt \pmod{27}.$$

Thus, using (7),

$$N \equiv 3B + 3rt \pmod{27} \Longrightarrow \frac{N}{3} \equiv B + rt \pmod 9.$$

Also, $\frac{s(N)}{3} = s(B) + rt \equiv B + rt \pmod 9$. Hence, $\frac{N}{3} \equiv \frac{s(N)}{3} \pmod 9$, which, together with $27|s(N)$ implies $27|N$, again contradicting our assumption.

Thus the assumption is false, establishing the (modified) proposition.

Also solved by the proposer.

Casting Out Eights

B-447 *Based on the previous proposal.*

Is there an analogue of B-446 in base 5?

Solution by Paul S. Bruckman, Concord, CA

Given

(1) $$N = \sum_{k=0}^{m} a_k 5^k,$$

where the a_k's are digits in base 5 ($a_k = 0, 1, 2, 3,$ or 4), let the sum of the digits be given by

(2) $$s(N) = \sum_{k=0}^{m} a_k.$$

We note that $N - s(N) = \sum_{k=0}^{m} a_k(5^k - 1) \equiv 0 \pmod 4$, since $5^k \equiv 1 \pmod 4$. Thus

(3) $$4|N \text{ iff } 4|s(N).$$

The analogue suggested by B-446 would probably read as follows: If 8 divides the sum of the digits of N (in base 5), then 8 divides one of the integers formed by permuting the digits of N (in base 5).

Unfortunately, the above proposition is false, unless additional conditions on N are specified. A counterexample is

$$N = 3{,}908 = (111113)_5;$$

although $8|s(N) = 8$, we find that all six integers formed by permuting the digits (in base 5) of N are congruent to 4 (mod 8), and therefore not divisible by 8.

The following is the corrected (though more complicated) version of the proposition: If 8 divides the sum of the digits of N (in base 5), then 8 divides one of the integers formed by permuting the digits of N (in base 5), *unless* all the digits of N are odd (i.e., 1 or 3) and the number of such digits is congruent to 2 (mod 4). In the latter case, all the permutations are congruent to 4 (mod 8).

(The proof is similar to that of B-446 and was deleted by the Elementary Problems Editor.)

Sum of Products Modulo 5

B-448 *Proposed by Herta T. Freitag, Roanoke, Va*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} F_{5i+1}L_{5i} \equiv 0 \pmod 5.$$

Solution by John Ivie, Glendale, AZ

It suffices to show that each pair $F_{5i+1}L_{5i} + F_{5i+6}L_{5i+5i}$ in the summation is divisible by 5. Using the Binet Formula, this pair equals

$$F_{10i+1} + F_{10i+11} = 5L_{10i+6}.$$

Also solved by Paul S. Bruckman, Bob Prielipp, Charles B. Shields, Sahib Singh, Lawrence Somer, Charles R. Wall, Stephen Worotynec, Gregory Wulczyn, and the proposer.

Sum of Products Modulo 7

B-449 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} (-1)^{i+1}F_{8i+1}L_{8i} \equiv 0 \pmod 7.$$

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

It is easy to show that

$$(-1)^{i+1}F_{8i+1}L_{8i} = (-1)^{i+1}F_{16i+1} + (-1)^{i+1}.$$

The powers of -1 telescope, since the number of summands is even, and thus

$$\sum_{i=1}^{2t} (-1)^{i+1}F_{8i+1}L_{8i} = \sum_{i=1}^{2t} (-1)^{i+1}F_{16i+1}.$$

We group the $2t$ summands in the latter sum into t pairs, each of which is divisible by 21:

$$F_{16i+1} - F_{16i+17} = -F_8L_{16i+9} = -21L_{16i+9}.$$

The stronger result, that the original sum is divisible by 21 (rather than merely 7), follows at once.

Also solved by Paul S. Bruckman, John Ivie, Bob Prielipp, Sahib Singh, Lawrence Somer, Stephen Worotynec, Gregory Wulczyn, and the proposer.

Lucas Quadratic Residue

B-450 *Proposed by Lawrence Somer, Washington, D.C.*

Let the sequence $\{H_n\}_{n=0}^{\infty}$ be defined by $H_n = F_{2n} + F_{2n+2}$.

(a) Show that 5 is a quadratic residue modulo H_n for $n \geq 0$.
(b) Does H_n satisfy a recursion relation of the form $H_{n+2} = cH_{n+1} + dH_n$, with c and d constants? If so, what is the relation?

Solution by E. Primrose, University of Leicester, England

We prove (b) first, and use it to prove (a).

(b) Examination of the first few terms suggests that

$$H_{n+2} = 3H_{n+1} - H_n,$$

and this is easily verified by using the defining relation for H_n and the recurrence relation for F_n.

(a) We prove that $H_{n+1}^2 - H_nH_{n+2} = 5$, which gives the required result. Now

$$\begin{aligned} H_{n+1}^2 - H_nH_{n+2} &= H_{n+1}^2 - H_n(3H_{n+1} - H_n) \\ &= H_n^2 + H_{n+1}(H_{n+1} - 3H_n) = H_n^2 - H_{n-1}H_{n+1}. \end{aligned}$$

It follows by induction that $H_{n+1}^2 - H_nH_{n+2} = H_1^2 - H_0H_2 = 5$.

Also solved by Paul S. Bruckman, Herta T. Freitag, John Ivie, John W. Milsom, Sahib Singh, Bob Prielipp, A.G. Shannon, Charles R. Wall, Gregory Wulczyn, and the proposer.

Consequence of the Euler-Fermat Theorem

B-451 *Proposed by Keats A. Pullen, Jr., Kingsville, MD*

Let k, m, and p be positive integers with p an odd prime. Show that in base $2p$ the units digits of $m^{k(p-1)+1}$ is the same as the units digit of m.

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

Let ϕ be Euler's function. Since p is an odd prime, $\phi(2p) = p - 1$ and therefore by Euler's Theorem, $m^{p-1} \equiv 1 \pmod{2p}$. We take the kth power of both sides and then multiply both sides by m to obtain

$$m^{k(p-1)+1} \equiv m \pmod{2p}$$

as asserted. [This assumes $\gcd(m, p) = 1$ but also follows for $p|m$.]

Also solved by Paul S. Bruckman, Herta T. Freitag, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

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