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A GENERALIZATION OF THE GOLDEN SECTION

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Introduction

It may surprise some people to find that the name "golden section," or, more precisely, *goldener Schnitt*, for the division of a line AB at a point C such that $AB \cdot CB = AC^2$, seems to appear in print for the first time in 1835 in the book *Die reine Elementar-Mathematik* by Martin Ohm, the younger brother of the physicist Georg Simon Ohm. By 1849, it had reached the title of a book: *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harminischen Theilung* by A. Wiegang. The first use in English appears to have been in the ninth edition of the *Encyclopaedia Britannica* (1875), in an article on Aesthetics by James Sully, in which he refers to the "interesting experimental enquiry . . . instituted by Fechner into the alleged superiority of 'the golden section' as a visible proportion. Zeising, the author of this theory, asserts that the most pleasing division of a line, say in a cross, is the golden section . . . ." The first English use in a purely mathematical context appears to be in G. Chrystal's *Introduction to Algebra* (1898).

The question of when the name first appeared, in any language, was raised by G. Sarton [11] in 1951, who specifically asked if any medieval references are known. The *Oxford English Dictionary* extends Sarton's list of names and references and, by implication, answers this question in the negative. (The 1933 edition of the *OED* is a reissue of the *New English Dictionary*, which appeared in parts between 1897 and 1928, together with a Supplement. The main dictionary entry "Golden," in a volume which appeared in 1900, makes no reference to the golden section, though it does cite mathematical references that will be noted later; the entry "Section" (1910) contains a reference to "medial section" (Leslie, *Elementary Geometry and Plane Trigonometry*, fourth edition, 1820) and to Chrystal's use of "golden section" noted above. The 1933

Supplement does not appear to contain any further references. A further Supplement, which started publication in 1972, has a long and detailed entry under "Golden" which is clearly based on and extends, but does not answer, Sarton's question.) Among the other names are: the Italian *divina proportione* (Luca Pacioli, in his book of that name, published in Venice in 1509) or Latin *proportio divina* (in a letter from Johannes Kepler to Joachim Tanck on May 12, 1608; then in Kepler's book *De Nive Sexangula*, 1611); the golden medial; the medial section; and the golden mean. This last term "golden mean" is credited by the *OED* to D'Arcy W. Thompson. (Further complications! The *OED*—1972 Supplement entry "Golden"—cites p. 643 of *On Growth and Form* [12]: "This celebrated series, which . . . is closely connected with the *Sectio aurea* or Golden Mean, is commonly called the Fibonacci series." The reference is to the now rare first edition of 1917; the second edition has an expanded and elaborately erudite version of this footnote on pp. 923 and 924, which starts differently: "This celebrated series corresponds to the continued fraction 1 + 1/1+ 1/1+ etc., [though Thompson, who uses a slightly different layout of the fraction, omits the first term in both versions of the footnote] and converges to 1.618..., the numerical equivalent of the *sectio divina*, or 'Golden Mean.'" This same dictionary entry later assigns the first use of the Latinized *sectio aurea* to J. Helemes, in 1844, in a heading in the *Archiv für Mathematik und Physik*, IV, 15: "*Eine . . . Auflösung der sectio aurea.*") Unfortunately, the same expression "golden mean" is usually applied to the Aristotelian principle of moderation: avoid extremes. Other quite different things with similar names are the golden rule (the rule of three; see the *OED* 1933 edition entry "Golden" for references) and the golden number (the astronomical index of Meton's lunar cycle of nineteen years). Also E. T. Bell, in "The Golden and Platinum Proportions" [2], refers to "the so-called golden proportion 6:9::8:12," but I cannot decide whether this article is meant as a serious contribution or not. If confusion and misapprehension were confined to nomenclature, that would, it is evident, be bad enough; alas, more is to be described, after a paragraph of sanity.

The mathematical theory of the golden section can be found in many places. I would cite Chapter 11 of H. S. M. Coxeter's *Introduction to Geometry* [4] as both the best and most accessible reference, and further developments can be found in other of Coxeter's works. The briefest acquaintance with any treatment of the Fibonacci series will indicate why many accounts of that topic will tend to the golden section, and *The Fibonacci Quarterly* is a rich source of articles and references on this subject. That there appears to be a connection between the Fibonacci numbers (and hence the golden section) and phyllotaxis (i.e., the arrangement of leaves on a stem, scales on a pine cone, florets on a sunflower, infloresences on a cauliflower, etc.) is an old and tantalizing observation. The subject is introduced in Coxeter [4], a brief historical survey is included in a comprehensive paper by Adler [1], and Coxeter [5] gives a short and authoritative statement.

The application of the golden section to other fields has, however, created a vast and generally romantic or unreliable literature. For instance, the application to aesthetics is, by its nature, subjective and controversial; a good brief survey with references is given in Wittkower [13]. For a comprehensive example of the genre, see the rival explanation and critical view of the role of the golden section in literature, art, and architecture in Brunés, *The Secrets of Ancient Geometry* [3]. (Lest I be incorrectly understood to be dismissing the scientific and experimental study of aesthetics as worthless, let me cite H. L. F. von Helmhlotz's *On the Sensations of Tone* [10] as an impressively successful example of this type of investigation, the very acme of science, mathematics, scholarship, and sensibility. In particular this book contains the first explanation of the ancient Greek observation that harmony seems to be connected with small integral ratios. But it is precisely Helmholtz's masterly blend of acoustics, physiology, physics, and mathematics that establishes firmly a standard which so few other writers on scientific aesthetice approach.)

With this outline of the recent history of the golden section behind us, my objective here is to treat the construction as it is described in Euclid's *Elements* under the name of "the line divided in extreme and mean ratio" and to develop and explore beyond the propositions we find proved there. My covert purpose is historical: to pose implicitly the question of whether the generalizations to be described here might have had any part, now lost, in the development of early Greek mathematics. To isolate this discussion of the ancient period from the later convoluted ramifications sketched in this introduction, I would like to finish with what is, I hope, an accurate description of the surviving evidence about the Greek period: the propositions to be found in Euclid's *Elements* constitute the only direct, explicit, and unambiguous surviving references to the construction in early Greek mathematics, philosophy, and literature; and the only other surviving Greek references are to be found in mathematical contexts, in Ptolemy's *Syntaxis*, Pappus' *Collectio*, Hypsicles' "Book XIV" of the *Elements*, and an anonymous Scholion on Book II of the *Elements*.

Acknowledgments

I would like to thank I. Adler, M. Brown, H. Cherniss, H. S. M. Coxeter, R. Fischler, S. Fowler, T. O. Hawkes, S. J. Patterson, E. Shiels, D. T. Whiteside, C. Wilson, and many members of Marlboro College, Vermont, for diverse kinds of help during the preparation of this article.

The Definition in Euclid's *Elements*

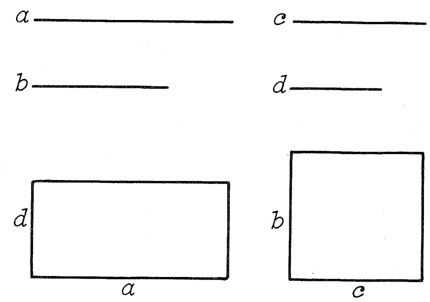
The golden ratio is defined at the beginning of Book VI of the *Elements*:

> A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.

Book VI applies the abstract proportion theory of Book V to geometrical magnitudes, and Proposition 16 describes how to manipulate the proportion in the definition above into a geometrical statement:

> If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes is equal to the rectangle contained by the means, the four straight lines will be proportional.

Otherwise said, if a, b, c, and d are four lines such that $a{:}b{::}c{:}d$, then rectangle (a, d) = rectangle (b, c) and conversely. Hence, if C divides the line AB in the golden section, the rectangle with sides AB and BC is equal to the square with side AC.



This is meant literally. An elaborate theory, now generally called the "application of areas," is developed in the *Elements*, and this describes how, for example, to manipulate any rectilinear plane area into another area equal to the original area and similar to a third figure. Our arithmetical definition of area ("base × height") is not needed and is never used; indeed, this theory of application of areas, together with the Book V theory of proportions, provides a completely adequate alternative to the construction of the real numbers and their use in plane rectilinear geometry. It merits considerable respect, and gets it: the same (probably equally unreliable) story is found about Pythagoras sacrificing an ox to the discovery of a result on the application of areas as is also told about the theorem on right angle triangles.

The golden section is constructed in Proposition 30:

> To cut a given finite straight line in extreme and mean ratio,

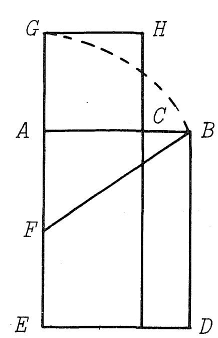
and the method used there involves an elaboration of the theory called "application with excess." Fortunately, an easier construction is possible and has already been given in Book II; and the manuscripts that we possess of the *Elements* contain a second, possibly interpolated, proof of VI, 30, referring back to this earlier construction. Using this method, it is possible to bypass the use of proportion theory, and the elaborations of the theory of application of areas, and to give a direct definition and construction of the golden section. This is now we shall proceed.

The Construction of the Line Divided in Extreme and Mean Ratio, and Its Generalization

Book II, Proposition 11, describes how:

> To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment,

and we shall hereinafter adopt this as the definition of the extreme and mean ratio. The construction is straightforward:



To construct the required point C on AB, draw the square $ABDE$; take F to be the midpoint of AE, and G on EA produced such that $FG = FB$. If $ACHG$ is the square with side AG, then C cuts AB in mean and extreme ratio.

The verification of this is easy:

$$\begin{aligned} FG^2 &= (AF + AG)^2 \\ &= AF^2 + AC^2 + 2AF \cdot AC \quad \text{(Since } AG = AC.\text{)} \end{aligned}$$

But $\qquad FG^2 = FB^2 = AF^2 + AB^2$. (By Pythagoras' theorem.)

Therefore, $AC^2 + 2AF \cdot AG = AB^2$.
Subtract $2AF \cdot AG = AE \cdot AC$ from both sides,
then $AC^2 = AB \cdot CB = CB \cdot BD$.

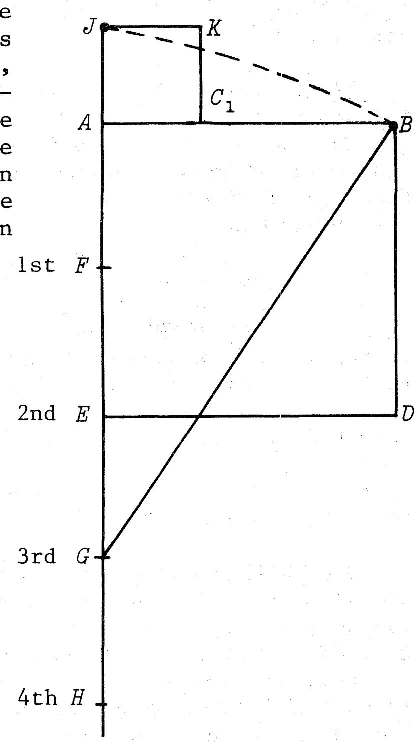
Q.E.F.

This proof can be read as if $AF \cdot AC$, for example, represented the product of two numbers, the lengths of AF and AC; or the purist can interpret $AF \cdot AC$ as a rectangle with sides equal to the lines AF and AC and, using some obvious manipulations, check that the proof makes sense and is correct. This latter method is in the spirit of the techniques of application of areas, though none of the subtle manipulations of that theory are needed.

It is clear that it must be the rectangle contained by the whole and the *lesser* segment that will be equal to the square on the *greater* segment, since the square on the lesser segment will fit inside the rectangle contained by the whole and the greater segment and so it has smaller area. (The common notions at the beginning of Book I set out what are, in effect, the axioms of a theory of equality and inequality of area or, more strictly, of content; and Common Notion 5 states: The whole is greater than the part.)

We now describe the generalization that we call the *nth order extreme and mean ratio*, abbreviated to the *noem ratio*. There is one such construction for each integer n, and the golden section corresponds to the case $n = 1$; the implications of the construction are somewhat simpler for the case of even values of n, and therefore we shall always illustrate the case of $n = 3$; and we shall shortly introduce and use a consistent and general notation and terminology to describe the resulting configuration.

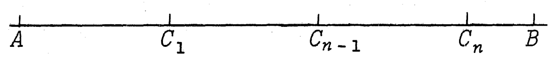
Start with the square $ABDE$ on the given line AB, and on AE produced as necessary, take points F, G, H, as shown, with $\frac{1}{2}AB = AF = FE = EG = GH$, etc.; then these points will be used in the construction of the 1st, 2nd, 3rd, 4th, etc., extreme and mean ratios. We always illustrate the case of $n = 3$ and so, here, work from the point G. On EA produced, take J such that $GJ = GB$; then the square $AJKC_1$ defines the point C_1 dividing AB in the 3rd extreme and mean ratio.



The Definition and Properties of the Noem Ratio

We start with the basic defining property of the generalization, and show that it is possessed by our constructed point.

Definition: The point C_1 is said to divide AB in the *noem ratio* (read: nth-order extreme and mean ratio) if, taking points C_2, ..., C_{n-1}, C_n on AB such that

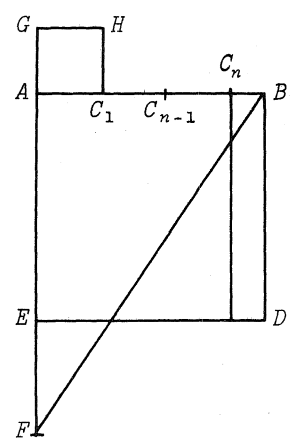


$AC_1 = C_1C_2 = \cdots = C_{n-1}C_n$, then C_n lies between A and B and $AB \cdot C_nB = AC_1^2$.

Note that the latter condition implies that C_nB is less than AC_1; it will be called the "lesser segment" of the noem ratio. The greater segment of the golden ratio generalizes two ways: to AC_1, which we call the initial segment of the noem ratio; and to AC_n, which we again call the greater segment of the noem ratio. Care must be exercised in generalizing the results on the golden section to make the appropriate choice. As remarked earlier, we shall always illustrate the case of $n = 3$, and will always use the same letters to label the points, calling the three division points C_1, C_{n-1}, C_n, so that their roles will be clear. Proofs will be given for the general case, sometimes referring to a phantom point C_2 and adding a few dots "+ ⋯ +."

Proposition: The point C_1, described in the construction, divides AB in the noem ratio.

Proof: The figure illustrates the construction for the case $n = 3$. The proof is a straightforward generalization of the proof given in the case $n = 1$, and we can even use the same letters to identify the vertices of the figure.



As before,

$$FG^2 = (AF + AG)^2$$
$$= AF^2 + AC_1^2 + 2AF \cdot AC_1.$$

But $\quad FG^2 = FB^2 = AF^2 + AB^2.$

Therefore, $AC_1^2 + 2AF \cdot AC_1 = AB^2.$

But $\quad 2AF \cdot AC_1 = nAE \cdot AC_1 = AE \cdot AC_n.$

$\left(\text{Since } AF = \frac{n}{2}AE, \text{ and } nAC_1 = AC_n.\right)$

Hence $AC_n < AB$ and, subtracting $AE \cdot AC_n$ from both sides, we see that $AC_1^2 = AB \cdot C_nB$. Q.E.F.

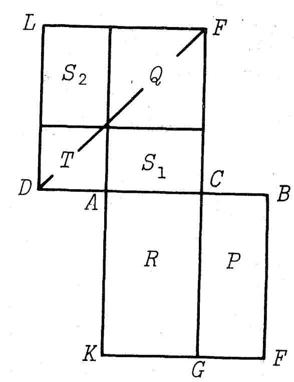
Book XIII of Euclid's *Elements* contains the details of the construction of the five regular "Platonic" solids, and a proof that these are the only regular solids; but it contains a lot more material besides that. In particular, it starts with six propositions on the extreme and mean ratio, together with alternative proofs of these results illustrating a method of "analysis and synthesis." These propositions follow on in the style of Book II—in particular, they do not explicitly need to use any more than the rudiments of the theory of application of areas—and they can easily be generalized to apply to the noem ratios. We now alternate the enunciations of these Euclidean propositions with their generalizations, interposing some general remarks. (Later propositions of Book XIII describe relationships between the extreme and mean ratio and pentagons, hexagons, decagons, icosahedra, and dodecahedra; we shall not consider them here.)

> XIII, Proposition 1. If a straight line be cut in extreme and mean ratio, the square on the greater segment added to half of the whole is five times the square on the half.

Paraphrase of Euclid's Proof:

If AB is cut in extreme and mean ratio at C, and $DA = \frac{1}{2}AB$, then we prove $CD^2 = 5AD^2$.

Draw the squares on DC and AB, and complete the figure as shown. (In addition to the Euclidean labelling of the vertices, we have also labelled the regions of the figure.)



We know that $AB \cdot CB = AC^2$ (Definition of mean and extreme ratio)

i.e., $P = Q$,

and $AB \cdot AC = 2AD \cdot AC$ (Since $AB = 2AD$)

i.e., $R = 2S_1 = S_1 + S_2$;

hence $P + R = Q + S_1 + S_2$.

Adding $AD^2 = T$, and assembling the result into squares,

$$DC^2 = AB^2 + AD^2.$$

But $AB^2 = 4AD^2$, so $DC^2 = 5AD^2$. Q.E.D.

Remark: Our way, today, of considering the golden ratio is almost always to identify it with the real number $\frac{1}{2}(\sqrt{5}+1)$; this and the following propositions represent the closest approach we find in surviving Greek texts to this evaluation. For instance, this proposition implies that if $AB = 2$, then $CD = \sqrt{5}$ (i.e., the side of a square of area equal to the rectangle with sides AB and $5AB$) so $AC = \sqrt{5} - 1$, and the ratio is $2:(\sqrt{5} - 1)$ [$= \frac{1}{2}(\sqrt{5} + 1):1$].

Proposition 1': If a straight line be cut in the noem ratio, the square on the initial segment added to n times half of the whole is $n^2 + 4$ times the square on the half.

Remark: It is standard Euclidean practice to handle such a general proof by choosing a particular small value of n, typically $n = 2, 3,$ or 4. The figures for our proofs differ very slightly according as n is even or odd (a consequence of the occurrence of halves in the construction), with the case of even

n being slightly simpler. It is also standard Euclidean practice when there are several different cases to a proposition only to consider the most complicated one. Therefore, our choice of $n = 3$ is in line with the Euclidean procedure. We shall, however, use a general labelling system, writing C_1, C_{n-1}, C_n, rather than C_1, C_2, C_3, and develop further the practice of labelling regions of the figure, using letters P, Q, R, etc., and suffixing to denote equal regions, so $P_1 = P_2 = P_3$ etc. Euclidean practice appears to be to label only the vertices of the figure, working through the alphabet strictly in order of occurrence in the setting-out and construction of the figure.

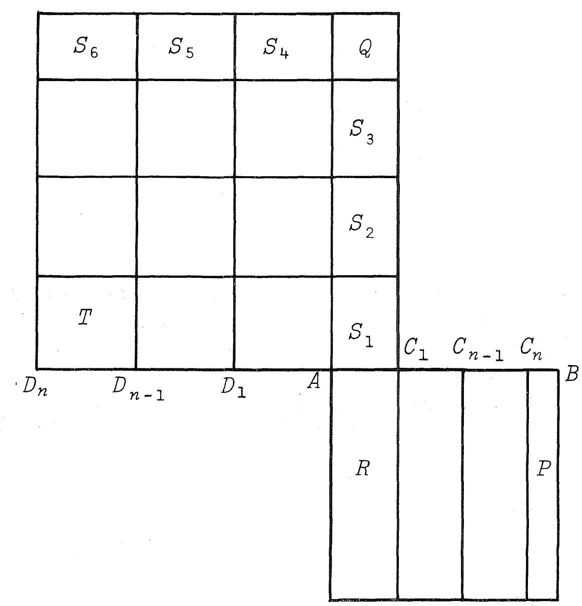
A final point in which our enunciation differs from Euclidean practice is in referring to the chosen parameter n. A more idiomatic expression, as rendered in English, might read:

Proposition 1'': If a straight line be cut in the general extreme and mean ratio to some number, the square on the initial segment added to that number of segments each equal to half of the whole is the square of that number increased by four times the square on the half.

Purists might like to try a similar rephrasing of later generalizations!

Proof: If AB is cut in the noem ratio at C_1, and $D_nA = \frac{n}{2}AB$, then we prove that $D_nC_1^2 = (n^2 + 4)AD_1^2$.

Draw the squares on D_nC_1 and AB, and complete the figure as shown:



We know that $AB \cdot C_nB = AC_1^2$ (Definition of the noem ratio)

i.e., $P = Q$,

and $AB \cdot AC_1 = 2AD_1 \cdot AC_1$ (Since $AB = 2AD_1$)

i.e., $R = 2S_1$,

and hence, $$P + nR = Q + 2nS_1.$$

Adding $AD_n^2 = n^2T$ and assembling the result into squares, we get

$$\begin{aligned} D_nC_1^2 &= AB^2 + n^2T \\ &= (n^2+4)T \qquad \text{(Since } AB^2 = 4T\text{)} \\ &= (n^2+4)AD_1^2. \end{aligned}$$

Q.E.D.

The next propositions give the converses to these results. We start with Euclid's enunciation:

> XIII, Proposition 2. If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

The Euclidean practice of never referring to a particular figure can make the enunciations of propositions very cumbersome, and these propositions, together with the propositions of Book II contain some particularly awkward examples. In these cases, it is best to ignore the enunciation and proceed directly into Euclid's proof of the proposition, where the setting-out will give a more accessible explanation of the result. In this case we find, paraphrasing and adjusting the labelling to accord with our convention, that if C and B are taken on a line DA produced with $DC^2 = 5DA^2$ and $AB = 2DA$, then C cuts AB in the extreme and mean ratio with AC the greater segment.

Proposition 2': If a line D_nA is divided equally into

$$D_nD_{n-1} = D_{n-1}D_{n-2} = \cdots = D_1A,$$

and C_1 and B are taken on D_nA produced with

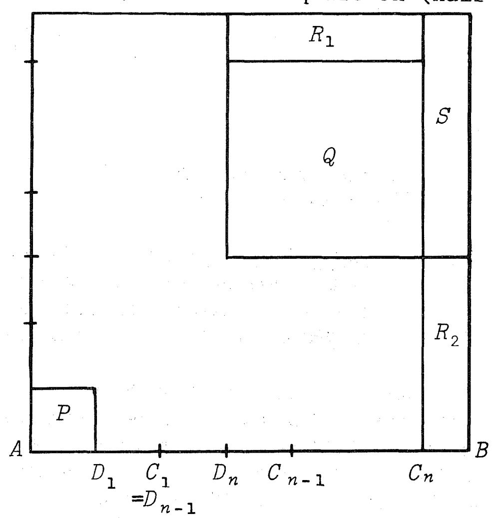
$$D_nC_1^2 = (n^2+4)D_nD_{n-1}^2 \quad \text{and} \quad AB = \frac{2}{n}D_nA = 2D_nD_{n-1},$$

then C_1 cuts AB in the noem ratio with AC_1 the initial segment.

Proof: For both propositions we can construct the same figures as for the preceding propositions and then read the previous arguments backwards. Q.E.D.

> XIII, Proposition 3. If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to half of the greater segment is five times the square on half of the greater segment.

Proposition 3': If C_1 cuts AB in the noem ratio, and $AD_n = \frac{n}{2}AC_1$ (as shown), then $D_nB^2 = (n^2 + 4)AD_1^2$; i.e., the square on (the lesser segment C_nB added to half of the greater segment AC_n) is equal to $(n^2 + 4)$ times the square on (half of the initial segment AC_1).



Proof: Construct the figure shown, where A, B, C_1, C_{n-1}, C_n are as usual, and $AD_1 = D_1D_{n-1} = D_{n-1}D_n = \frac{1}{2}AC_1$.

First observe that

$$\begin{aligned} D_nC_n &= AC_n - AD_n \\ &= nAC_1 - nAD_1 \\ &= nAD_1 \quad (\text{Since } AC_1 = 2AD_1) \\ &= AD_n. \end{aligned}$$

Hence,

$$\begin{aligned} D_nB^2 &= Q + R_1 + S \\ &= Q + R_2 + S \\ &= Q + AB \cdot C_nB \\ &= Q + AC_1^2 \quad (\text{Since } C_1 \text{ divides } AB \text{ in the noem ratio.}) \\ &= (n^2 + 4)AD_1^2. \quad (\text{Since } Q = D_nC_n^2 = n^2AD_1^2 \text{ and } AC_1 = 2AD_1.) \end{aligned}$$

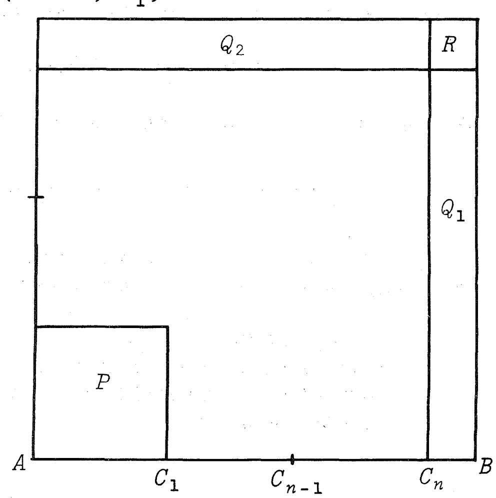
Q.E.D.

XIII, Proposition 4. If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Proposition 4': Let AB be cut in the noem ratio at C_1, then

$$AB^2 + C_nB^2 = (n^2 + 2)AC_1^2,$$

i.e., the square on the whole and the square on the lesser segment together are $(n^2 + 2)$ times the square on the initial segment.



Proof: We have that $C_nB \cdot AB = AC_1^2$,

i.e., $Q_1 + R = P$.

Hence, $Q_1 + R + Q_2 + R = 2P$.

Adding $AC_n^2 = n^2P$ to each side, and assembling into squares, we see that $AB^2 + C_nB^2 = (n^2 + 2)AC_1^2$. Q.E.D.

> XIII, Proposition 5. If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in mean and extreme ratio, and the original straight line is the greater segment.

Proposition 5': If C_1 cuts AB in the noem ratio, and A_n is taken on BA produced with $A_nB = nAB$, and D on BA_n produced with $DA_n = AC_1$, then DB is cut in the noem ratio by A.

Note: In the Euclidean proposition, $n = 1$ and $A_n = A$; therefore, there is no need to mention the first step of constructing the point A_n. After this step the generalization states that, if there be added to A_nB a line equal to the initial segment, the whole BD has then been cut in the noem ratio, with the line BA_n being the greater segment, and so the original line BA the initial segment.

Proof: Complete the figure as shown; we want to show that $DA_n \cdot DB = AB^2$. Now,

$$DA_n \cdot DA = P + Q_1 + Q_{n-1} + Q_n$$

and

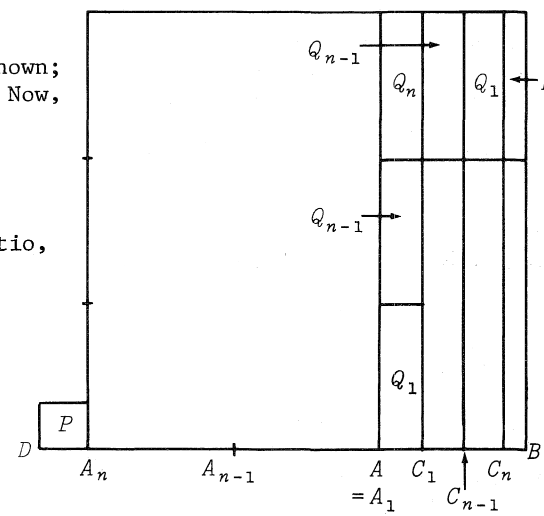
$$AB^2 = Q_n + Q_{n-1} + Q_1 + R$$

and, since C_1 cuts AB in the noem ratio,

$$AB \cdot C_nB = AC_1^2,$$

i.e., $$P = R.$$

Hence $$DA_n \cdot DB = AB^2.$$



> XIII, Proposition 6. If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome.

This result, together with its proof, generalizes directly to the noem ratio, but an explanation of what it means depends on a knowledge of the long and difficult Book X. It is perhaps worth noting that Euclid uses the words "rational" and "irrational" here in completely different sense from our modern usage: a short, though oversimplified explanation is that when a unit line ρ has been chosen, then anything of the form $\sqrt{\frac{p}{q}} \cdot \rho$ (where p and q are integers) is called rational; anything not of that form is an irrational; and an apotome is an irrational line that can be expressed as a difference of two rational lines, $\sqrt{\frac{p}{q}} \cdot \rho - \sqrt{\frac{r}{s}} \cdot \rho$.

Ratio in Eudlid's *Elements*

It is a curious and remarkable fact that ratio is not defined either in Eudlid's *Elements*, or anywhere else in the surviving corpus of Greek mathematics. All that we have is a vague description of the word at Book V, Definition 3:

> A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

What *is* defined (at Book V, Definition 5) is proportion, which is a relation that may or may not hold among four magnitudes, $a:b::c:d$; and we can think of it, and appear to be encouraged in this by Euclid, in terms of the equality of two "ratios." An examination of the scanty surviving evidence of pre-Eudlidean mathematics, and a reinterpretation of some of the books of the *Elements* has led me to suggest that ratio might have been defined, in the period before the development of the abstract proportion theory that we find in Book V of the *Elements*, by a process based on the "Euclidean" subtraction algorithm. (Actually, what little evidence we have indicates that the person who realized the importance of the procedure might have been Theaetetus, a colleague and friend of Plato, so the "Theaetetan subtraction algorism" might be a more appropriate name; here, I have also corrected what the *OED* calls a "pseudo-etymological perversion . . . in which algorithm is learnedly confused with Greek αριθμός.") Let me illustrate this by describing the operation of the procedure on two lines a_0 and a_1. Suppose that a_1 goes into a_0 some number n_0 of times, leaving a remainder a_2 less than a_1; and then a_2 goes into a_1 some number n_1 of times, leaving a remainder a_3; etc. Then the ratio $a_0:a_1$ will be *defined* by the sequence of integers $[n_0, n_1, n_2, \ldots]$.

If, at any stage, a remainder is zero, the process terminates, and this is characteristic of commensurable ratios. Among incommensurable ratios, with nonterminating expansions, the simplest will be the ratio in which, at each step, the smaller magnitude goes once into the larger magnitude, leaving a remainder for the next step, thus giving the ratio $[1, 1, 1, \ldots]$. This is the golden ratio, as can immediately be deduced from the figure of the regular pentagon of which the diagonals, which form an inscribed pentagon, cut each other in the golden ratio (this is explicitly proved at XIII, 8, but the result is implicit in the construction of the pentagon given at IV, 11); or it can easily be deduced from the defining property of the ratio. What we have been constructing here are the next simplest incommensurable ratios, of the form $[n, n, n, \ldots]$, in which, at each stage, the smaller magnitude goes n times with a remainder into the larger magnitude. By using a bit of algebra we can easily work out the numerical value θ of this ratio, since

$$\theta = n + \cfrac{1}{n + \cfrac{1}{n + \ldots}} = n + \frac{1}{\theta},$$

so $\theta^2 - n\theta - 1 = 0$, and, taking the positive root,

$$\theta = \tfrac{1}{2}(\sqrt{(n^2 + 4)} + n).$$

[Alternatively, we can read off from the construction that

$$\theta = AB/AC_1 = 2/(\sqrt{(n^2 + 4)} - n) = \tfrac{1}{2}(\sqrt{(n^2 + 4)} + n).]$$

This explains the occurrence of the number 5, generalizing to $n^2 + 4$, the halves, and the addition and subtraction of segments in the propositions that we have been proving.

It is possible to extend the construction, and thus describe a procedure for constructing any ratio that eventually becomes periodic, though the longer the period, the more involved becomes a perliminary calculation of two parameters needed in the construction. (One of these parameters describes the location of the initial point on the left-hand edge of the square on AB, in our diagram; the other describes the position of an auxiliary point B' on AB; the construction then continues from these two points as before.) Further details of these constructions, together with details of the historical and mathematical ideas that fill out, explain, and set in context these remarks, are given in the papers [7], [8], and [9].

I do not know whether any of the noem ratios, with $n \geq 3$, occur in any regular or semiregular figure, generalizing the appearance of the golden section in the pentagon and other figures, and the ratio [1, 2, 2, 2, ...] of the diagonal and side of a square.

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