THE FIBONACCI SEQUENCE IN SUCCESSIVE PARTITIONS OF A GOLDEN TRIANGLE

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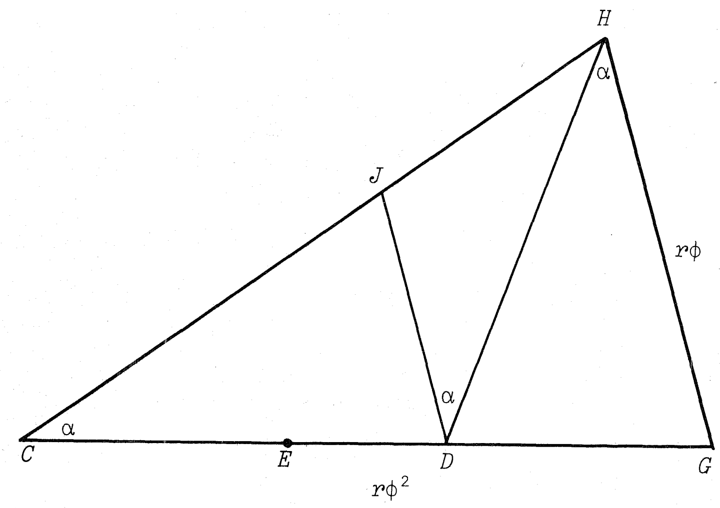
A Golden Triangle is a triangle with two of its sides in the ratio $\phi:1$, where ϕ is the Fibonacci Ratio, i.e., $\phi = \frac{1}{2}(1 + \sqrt{5}) \cong 1.618$. Let ΔABC be a triangle whose sides are a, b, and c and let $a/b = k > 1$. Bicknell and Hoggatt [1] have shown that (1) a triangle with a side equal to b can be removed from ΔABC to leave a triangle similar to ΔABC if and only if $k = \phi$, and (2) a triangle similar to ΔABC can be removed from ΔABC to leave a triangle such that the ratios of the areas of ΔABC and the triangle remaining is k if and only if $k = \phi$.

Unlike the Golden Rectangle whose adjacent sides are in the ratio $\phi:1$ (or $1:\phi$), the Golden Triangle does not have a single shape. The diagonal of a Golden Rectangle divides it into two Golden Triangles whose sides are in the ratio $1:\phi:\sqrt{\phi^2 + 1}$. The most celebrated Golden Triangle, which can be found in the regular pentagon and regular decagon, has angles of 36°, 72°, and 72° and sides in the ratio $1:\phi:\phi$. In general, Bicknell and Hoggatt demonstrated that a Golden Triangle can be constructed with sides in the ratio $1:\phi:G$, where $\phi^{-1} < G < \phi^2$. Figure 1, adapted from their presentation, shows Golden Triangle CGH. Line GH is constructed to be of length $r\phi$ $(r > 0)$ and line CG to be of length $r\phi^2$. Line CG is twice divided in the Golden Section by points E and D, with $CE = DG = r$ and $ED = r/\phi$. A Golden Triangle is formed whenever H is a point on the circle whose center is G and whose radius is EG. Line DH produces $\Delta DGH \sim \Delta CGH$, and ΔCDH whose area is $1/\phi$ times the area of ΔCGH. In general, ΔCDH is not similar to ΔCGH. Nonetheless, ΔCDH is also a Golden Triangle, as $CH/DH = \phi$ [1].

The present paper will explore the consequences of successively partitioning Golden Triangles. To begin, let us show that ΔCDH can be partitioned into two triangles, one similar to itself and the other having an area $1/\phi$ times its own area. If line DJ is drawn parallel to line GH, one can readily verify that ΔDHJ is similar to ΔCDH. (Alternatively, we could have chosen point J so that $CH/CJ = \phi$. Lines DJ and GH would then be parallel, because $CH/CJ = CG/CD$.) We now need to show that the ratio of the area of ΔCDH to the area of ΔCDJ is ϕ. If we designate the area of ΔCGH by S, the area of ΔCDH is S/ϕ [1]. Since DJ is parallel to GH, $\Delta CDJ \sim \Delta CGH$. The ratio $CG/CD = \phi$, hence the area of ΔCDJ is S/ϕ^2. Accordingly, the ratio of ΔCDH to ΔCDJ is S/ϕ divided by S/ϕ^2, or ϕ. Since $S/\phi - S/\phi^2 = S/\phi^3$, we find that the area of ΔDHJ is S/ϕ^3.

We can note several other relationships. Two additional Golden Triangles, ΔCDJ and ΔDHJ, are produced so that ΔCGH is partitioned into three mutually exclusive Golden Triangles. Moreover, ΔCDJ is congruent to ΔDGH. They are similar, as both are similar to ΔCGH and both have areas equal to S/ϕ^2.

Moving beyond the Bicknell-Hoggatt demonstration and its immediate implications, we can show how successive partitions of Golden Triangles generate Fibonacci sequences. Let us repeat the above partitioning, subdividing all of the larger triangles produced in the previous partition. The partitions can be carried out in a manner analogous to the way in which ΔCDH was partitioned.



$$\phi = \frac{CG}{GH} = \frac{CH}{DH} = \frac{GH}{DG} = \frac{CD}{DJ} = \frac{DH}{HJ}$$

$$\alpha = \angle DCH = \angle DHG = \angle HDJ$$

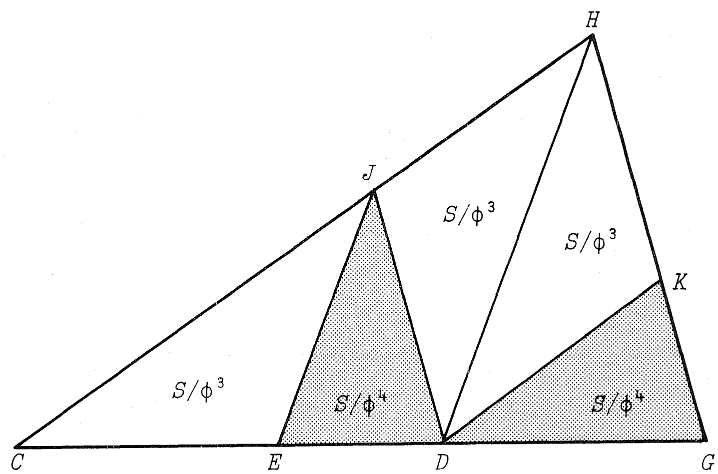
$$\Delta CDJ \cong \Delta DGH$$

FIGURE 1. The General Golden Triangle

For example, ΔCDJ can be split into two Golden Triangles by a line through point J parallel to HD. The resultant line is JE, which has a length equal to DH/ϕ and divides line CD in the Golden Section. As we proceed, the number of triangles and their areas are as follows:

Partition Number (n)	Fibonacci Number (F)	Number of Triangles	Area of Triangles
0	---	1	S
1	1	2	S/ϕ, S/ϕ^2
2	1	3	S/ϕ^2, S/ϕ^2, S/ϕ^3
3	2	5	S/ϕ^3, S/ϕ^3, S/ϕ^3, S/ϕ^4, S/ϕ^4
4	3	8	S/ϕ^4 (5 triangles), S/ϕ^5 (3 triangles)
5	5	13	S/ϕ^5 (8 triangles), S/ϕ^6 (5 triangles)
⋮	⋮	⋮	⋮
n	F_n	F_{n+2}	S/ϕ^n (F_{n+1} triangles), S/ϕ^{n+1} (F_n triangles)

The total number of triangles and the number of larger and smaller triangles increase in Fibonacci sequence. Figure 2 shows the five triangles produced by the third partition. The eight triangles produced by the fourth partition result from subdividing the three larger triangles in the figure. Because partitioning produced triangles whose areas, relative to the area of the partitioned triangle, are $1/\phi$ and $1/\phi^2$, the pattern is perpetuated.



S = Area ΔCGH

$\Delta CEJ \cong \Delta DHJ \cong \Delta DHK$

$\Delta DEJ \cong \Delta DGK$

FIGURE 2. The Partition of a Golden Triangle into Five Golden Triangles

By a repetition of the earlier demonstrations, it can be seen that every triangle that results from the partitioning is similar to one of the two Golden Triangles produced by the first partition (i.e., the partition effected by line DH), and that all triangles of the same area are congruent. Every triangle has an area equal to S/ϕ^i, for some integer i. For triangles similar to ΔCGH, i is even, while for triangles similar to ΔCDH, i is odd. Corresponding sides of triangles with areas S/ϕ^i and S/ϕ^{i+2} are in the ratio $\phi:1$. The total area of the larger triangles relative to the total area of the smaller is $\phi F_{n+1}/F_n$ after the nth partition, and that ratio approaches ϕ^2 as n becomes large.

Each partition illustrates the equation for powers of ϕ, i.e.,

(1) $$\phi^n = F_n\phi + F_{n-1}.$$

Dividing (1) through by ϕ^n, we have

(2) $$1 = \frac{F_n}{\phi^{n-1}} + \frac{F_{n-1}}{\phi^n},$$

which expresses the area of a unit Golden Triangle as the sum of the areas of partitioned triangles. For example, with $n = 4$, we have the situation before the fourth partition, shown in Figure 2, where

$$S = \frac{3S}{\phi^3} + \frac{2S}{\phi^4}. \tag{3}$$

Multiplying (3) by ϕ^4/S gives

$$\phi^4 = 3\phi + 2. \tag{4}$$

Let us move from the general case to the special case where the two Golden Triangles formed by the first partition are similar to ΔCGH, and hence to one another. In that case, the triangles must be "Fibonacci Right Triangles," with sides in the ratio $\phi:\phi^{3/2}:\phi^2$. To demonstrate that, consider triangles CDH and DGH in Figure 1. From (1) we know $\angle DCH = \angle DHG$. If triangles CDH and CGH are similar, $\angle CHD$ must equal $\angle DGH$ because $\angle CHD \neq \angle CHG$. Since $\Delta CDH \sim \Delta DGH$ and we have established equalities between two of their three angles, we must have $\angle CDH = \angle GDH$. As $\angle CDH$ and $\angle GDH$ sum to 180°, both of those angles equal 90° and line DH is an altitude. With $\angle CHG = \angle CHD + \angle DHG$ and $\angle DHG = \angle DCH$, we have $\angle CHG = \angle CHD + \angle DCH$. In right triangle CDH, $\angle CHD$ and $\angle DCH$ sum to 90°, hence $\angle CHG$ must be 90°. As Figure 1 was constructed with $GH = r\phi$ and $CG = r\phi^2$, applying the Pythagorean Theorem yields $r^2\phi^4 = r^2\phi^2 + CH^2$, and thus we find $CH = r\phi^{3/2}$.

The Fibonacci Right Triangle has been examined by a number of writers. Ghyka [2] identified it as one of the three most significant nonequilateral triangles. He noted that it was sometimes called the "Great Pyramid" triangle because its proportions are found in the Great Pyramid of Cheops, or the triangle of Price, after W. A. Price, who proved that it is the only right triangle whose sides are in geometric progression (i.e., if the sides of a triangle are 1, k, and k^2, $k = \sqrt{\phi}$ is the only positive real solution that satisfies the Pythagorean equation $1 + k^2 = k^4$). Hoggatt [3] noted that the altitude of a Fibonacci Right Triangle produced two Fibonacci Right Triangles that were "five parts congruent," that is, were similar and had two (but not three) sides of equal length. The Fibonacci Right Triangle is related to mean values, in that the harmonic, geometric, and arithmetic means of two positive numbers form a right triangle (the Fibonacci Right Triangle) if and only if those numbers are in the ratio $\phi^3:1$ [4]. In successive partitions of Fibonacci Right Triangles, all line segments are in Fibonacci proportions, as they are all multiples of $\phi^{i/2}$, with i an integer [5]. A multiply partitioned Fibonacci Right Triangle thus presents a striking geometric pattern. An example is given in Figure 3, which shows the 13 Fibonacci Right Triangles that result from five partitions of the original triangle.

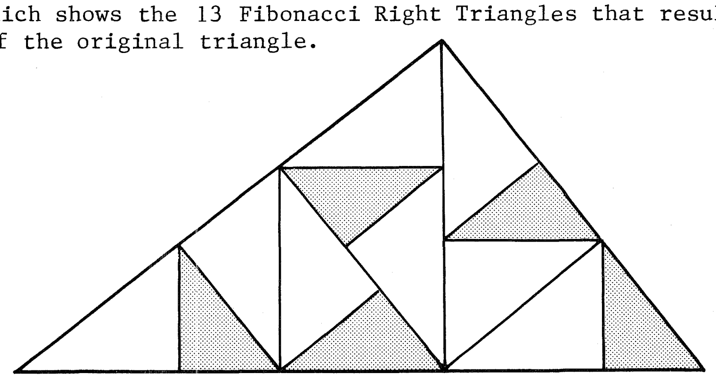


FIGURE 3. Five Partitions of a Fibonacci Right Triangle

In summary, successive partitions of a Golden Triangle provide a multifaceted geometric representation of the Fibonacci sequence. The triangles described above are Fibonacci in three different ways because they are in Fibonacci proportions with regard to their numbers, their areas, and the lengths of their sides. Golden Triangles not only embody the Fibonacci ratio, they also carry within them the ability to generate Fibonacci sequences.

References

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