DUCCI PROCESSES

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1. Introduction

During the 1930s Professor E. Ducci of Italy [1] defined a function whose domain and range are the set of quadruples of nonnegative integers. Let

$$f(x_1, x_2, x_3, x_4) = (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, |x_4 - x_1|).$$

Let $f^n(x_1, x_2, x_3, x_4)$ be the nth iteration of f. Ducci showed that for any choice of x_1, x_2, x_3, x_4 there exists an integer N such that

$$f^m(x_1, x_2, x_3, x_4) = (0, 0, 0, 0) \text{ for all } m > N.$$

We note the following properties of the function f of the previous paragraph:

(1) There exists a function $g(x, y)$ whose domain is the set of pairs of nonnegative integers and whose range is the set of nonnegative integers. [Here $g(x, y) = |x - y|$].

(2) $f(x_1, x_2, x_3, x_4) = (g(x_1, x_2), g(x_2, x_3), g(x_3, x_4), g(x_4, x_1))$.

(3) The four entries of $f^n(x_1, x_2, x_3, x_4)$ are bounded for all n. The bound depends on the initial choice of x_1, x_2, x_3, x_4.

We call the successive iterations of a function satisfying these conditions a Ducci process. Condition (3) guarantees that a Ducci process is either periodic or that after a finite number of steps (say N)

$$f^{n+1}(x_1, x_2, x_3, x_4) = f^n(x_1, x_2, x_3, x_4) \text{ for all } n > N.$$

If a function g generates a Ducci process of the latter type, we say that g is Ducci stable (or simply stable).

2. Illustrations

(1) Let $g(x, y) = \overline{x+y}$ (mod 3), where $\overline{x}$ (mod 3) is the least nonnegative integer congruent to x (mod 3). Then, an example shows that g is not stable. Set $x_1 = x_2 = x_3 = 0$ and $x_4 = 1$. We may tabulate the successive values of f as follows:

$$\begin{array}{ll}
 & (0,\ 0,\ 0,\ 1) \\
f^1: & (0,\ 0,\ 1,\ 1) \\
f^2: & (0,\ 1,\ 2,\ 1) \\
f^3: & (1,\ 0,\ 0,\ 1) \\
f^4: & (1,\ 0,\ 1,\ 2) \\
f^5: & (1,\ 1,\ 0,\ 0) \\
f^6: & (2,\ 1,\ 0,\ 1) \\
f^7: & (0,\ 1,\ 1,\ 0) \\
f^8: & (1,\ 2,\ 1,\ 0) \\
f^9: & (0,\ 0,\ 1,\ 1)
\end{array}$$

Since $f^9(0, 0, 0, 1) = f^1(0, 0, 0, 1) = (0, 0, 1, 1)$, the process is periodic with period 8 and g is not stable.

(2) Let $g(x, y) = \overline{x + y}$ (mod 8). We construct a similar table for the same initial values.

$$\begin{array}{ll} & (0, 0, 0, 1) \\ f^1: & (0, 0, 1, 1) \\ f^2: & (0, 1, 2, 1) \\ f^3: & (1, 3, 3, 1) \\ f^4: & (4, 6, 4, 2) \\ f^5: & (2, 2, 6, 6) \\ f^6: & (4, 0, 4, 0) \\ f^7: & (4, 4, 4, 4) \\ f^8: & (0, 0, 0, 0) \\ f^9: & (0, 0, 0, 0) \end{array}$$

We observe that for $n \geq 8$, $f^n(0, 0, 0, 1) = (0, 0, 0, 0)$. We prove below that g is stable, viz., that any choice of initial values leads to a similar result.

We now list a set of functions which can be proved to be stable. In some cases we prove the stability of the function and in others we leave the proof to the reader.

3. Theorem

The following functions are stable:

(1) $\overline{x + y}$ (mod 2^n), $n = 1, 2, 3, \ldots$.

(2) $\overline{x \cdot y}$ (mod 2^n), $n = 1, 2, 3, \ldots$.

(3) $\overline{x^t + y^t}$ (mod 2^n), $t = 2, 3, 4, \ldots$; $n = 1, 2, 3, \ldots$.

(4) $\overline{(x + y)^t}$ (mod 2^n), $t = 2, 3, 4, \ldots$; $n = 1, 2, 3, \ldots$.

(5) $|\overline{x^t - y^t}|$ (mod 2^n), $t = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$.

(6) $|\overline{(x - y)^t}|$ (mod 2^n), $t = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$.

(7) $\phi(x) + \phi(y)$, where ϕ is Euler's ϕ-function.

The notation $\overline{x}$ (mod 2^n) means the least nonnegative integer congruent to x modulo 2^n.

Proof of (1): We use f_i^n to denote the ith entry of the nth iteration of $f(x_1, x_2, x_3, x_4)$. The subscript $i + j$ of x will always represent $\overline{i + j}$ (mod 4). We first consider the function $g_1(x, y) = x + y$ and show that for any n:

$$f_i^{2(n+1)} = (2^n)[(2^n - 1)x_i + (2^n + 1)x_{i+2} + 2^n(x_{i+1} + x_{i+3})]. \tag{A}$$

We compute:
$$f_i^1 = x_i + x_{i+1}. \tag{B}$$

$$f_i^2 = x_i + 2x_{i+1} + x_{i+2}. \tag{C}$$

$$f_i^3 = x_i + 3x_{i+1} + 3x_{i+2} + x_{i+3}. \tag{D}$$

$$f_i^4 = 2x_i + 4x_{i+1} + 6x_{i+2} + 4x_{i+3}. \tag{E}$$

(A) is clearly true for $n = 1$ by (E). Suppose (A) is true for n. Then by (C)

$$f_i^{2(n+2)} = f_i^2(f_i^{2(n+1)}) = (2^{n+1})[(2^{n+1} - 1)x_{i+1} + (2^{n+1} + 1)x_{i+3} + (2^{n+1})(x_i + x_{i+2})]. \tag{F}$$

We note that for any iteration of $f(x_1, x_2, x_3, x_4)$ we can consider $(f_1^n, f_2^n, f_3^n, f_4^n)$ to be the same row as its transpositions $(f_4^n, f_1^n, f_2^n, f_3^n)$, $(f_3^n, f_4^n, f_1^n, f_2^n)$, and $(f_2^n, f_3^n, f_4^n, f_1^n)$. Therefore (F) indicates that (A) is also true for $n + 1$. It follows by finite induction that (A) is true for all n. Hence we conclude that

$$f_i^{2(n+1)} \equiv 0 \pmod{2^n},\ n = 1, 2, 3, \ldots .$$

Since $f^n(0, 0, 0, 0) = (0, 0, 0, 0)$ for all n, the stability of the function $g(x, y) = \overline{x + y} \pmod{2^n}$ is established.

Proof of (3): For any initial numbers x_1, x_2, x_3, x_4, there are six ways to arrange even and odd numbers:

(i)	(e, e, e, e)	(iv)	(e, b, e, b)
(ii)	(e, e, e, b)	(v)	(e, b, b, b)
(iii)	(e, e, b, b)	(vi)	(b, b, b, b)

where e and b represent even and odd numbers, respectively. Since the sum of the tth powers of two even (or two odd) numbers is even, the sum of the tth powers of an even number and an odd number is odd. Therefore, when we consider the function $g_2(x, y) = x^t + y^t$, the initial arrangements (ii) and (v) yield the following:

$$\begin{array}{llcl}
 & (e, e, e, b) & & (e, b, b, b) \\
f^1: & (e, e, b, b) & & (b, e, e, b) \\
f^2: & (e, b, e, b) & \text{and} & (b, e, b, e) \\
f^3: & (b, b, b, b) & & (b, b, b, b) \\
f^4: & (e, e, e, e) & & (e, e, e, e)
\end{array} \tag{G}$$

The arrangements (i), (iii), (iv), and (vi) are included in the above operations. Thus there exists an integer $m \leq 4$ such that all numbers of $f^m(x_1, x_2, x_3, x_4)$ are even numbers for the arrangements (i)-(vi).

Let $f^m(x_1, x_2, x_3, x_4) = (2^i m_1, 2^j m_2, 2^u m_3, 2^v m_4)$, where $i, j, u, v, m_1, m_2, m_3, m_4$ are positive integers. Without loss of generality, we may assume that $i \leq j, u, v$. Then we have

$$(2^i m_1)^t + (2^j m_2)^t = 2^{it} m_1^t + 2^{jt} m_2^t = 2^{it}(m_1^t + 2^{(j-i)t} m_2^t).$$

This indicates that the value of i in f^m will increase by at least t times at the next step (where $t \geq 2$). After a finite number of steps, we can obtain an integer q such that $f^q(x_1, x_2, x_3, x_4) = (2^h q_1, 2^\ell q_2, 2^r q_3, 2^s q_4)$, where $h, \ell, r, s, q_1, q_2, q_3, q_4$ are positive integers and $h, \ell, r, s \geq n$, i.e., all numbers of f^q are the multiples of 2^n. Thus, the four numbers of f^q are congruent to zero modulo 2^n. This shows that the function $g(x, y) = \overline{x^t + y^t} \pmod{2^n}$ is stable.

Before we prove the last statement of the theorem, let us recall a simple property of Euler's ϕ-function.

Lemma 1: For any even integer N, (i) if N is a power of 2, then $\phi(N) = (\frac{1}{2})N$; (ii) if N is not a power of 2, then $\phi(N) < (\frac{1}{2})N$.

Proof of (7): First, we consider only the initial numbers x_1, x_2, x_3, x_4 greater than 2. Since $\phi(x)$ is even for all $x > 2$. Therefore, we have

$$f^1(x_1, x_2, x_3, x_4) = (N_1, N_2, N_3, N_4),$$

where N_1, N_2, N_3, N_4 are even integers and $\min\{N_1, N_2, N_3, N_4\} \geq 4$. If $N_1 = N_2 = N_3 = N_4$, we can see below that statement (7) of the theorem is true in this case. If all four are not equal, by Lemma 1 it is clearly seen that the greatest integer (if two or three are equal and greater than the remaining, each of these may be called "the greatest") of $f^n(x_1, x_2, x_3, x_4)$ must get smaller within three steps for all n. Hence, after a finite number of steps (such as m), $f^m(x_1, x_2, x_3, x_4) = (N_5, N_6, N_7, N_8)$, where N_5, N_6, N_7, N_8 are even integers and either $N_5 = N_6 = N_7 = N_8 = 2^t$ for some integer $t \geq 3$, or $\max\{N_5, N_6, N_7, N_8\} = 4$. But, we also have $\min\{N_5, N_6, N_7, N_8\} \geq 4$ and

$$f^c(2^t, 2^t, 2^t, 2^t) = (2^t, 2^t, 2^t, 2^t) \text{ for all } c \text{ and } t.$$

This implies that the function $\phi(x) + \phi(y)$ is stable.

It remains only to show that the initial numbers x_1, x_2, x_3, x_4 contain some 2s or 1s [since $\phi(2) = \phi(1) = 1$, so we only need to consider either 1 or 2]. Suppose the initial numbers contain only one number 2, say $x_1 = 2$. Thus

$$f^1(x_1, x_2, x_3, x_4) = (1 + \phi(x_2), \phi(x_2) + \phi(x_3), \phi(x_3) + \phi(x_4), \phi(x_4) + 1).$$

Since $x_2, x_3, x_4 > 2$. Therefore, all four numbers of $f^1(x_1, x_2, x_3, x_4)$ are strictly greater than 2. Similarly, when the initial numbers contain two or three 2s, we can prove that there exists an integer $j \leq 3$ such that

$$f^j(x_1, x_2, x_3, x_4) = (J_1, J_2, J_3, J_4),$$

where J_1, J_2, J_3, J_4 are integers and $\min\{J_1, J_2, J_3, J_4\} > 2$. This completes the proof of (7).

4. Some More Ducci Processes

Let us denote the m-digit integer by

$$x = 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + 10a_2 + a_1$$

and

$$S_x^t = (a_m + a_{m-1} + \cdots + a_2 + a_1)^t, \; T_x^t = a_m^t + a_{m-1}^t + \cdots + a_2^t + a_1^t,$$

where $t = 1, 2, 3, \ldots$.

We now address the following problems:

(1) For what values of t is the function $|S_x^t - S_y^t|$ stable?

(2) For what values of t is the function $|T_x^t - T_y^t|$ stable?

(3) For what values of t and n is the function $\overline{T_x^t + T_y^t}$ (mod 2^n) stable?

Partial answers to these questions are given below.

Obviously, the function $|S_x^t - S_y^t|$ is stable for $t = 1$. In order to prove stability for $t = 2$, we need the following lemma.

Lemma 2: Let Z be the set of all nonnegative integers and let $H = \{3z: z \in Z\}$, $L = Z \backslash H$. Then for any $h, h_1, h_2 \in H$ and $\ell, \ell_1, \ell_2 \in L$ we have

(i) $|h_1^2 - h_2^2| \in H$ and $|\ell_1^2 - \ell_2^2| \in H$;

(ii) $|h^2 - \ell^2| \in L$.

Proof: For any $h \in H$, we have $h \equiv 0 \pmod 3$ and $h^2 \equiv 0 \pmod 3$. For any $\ell \in L$, we have either $\ell \equiv 1 \pmod 3$ or $\ell \equiv 2 \pmod 3$. But we see that $\ell^2 \equiv 1 \pmod 3$ for both cases. Therefore, we obtain

(i) $|h_1^2 - h_2^2| \equiv 0 \pmod 3$ and $|\ell_1^2 - \ell_2^2| \equiv 0 \pmod 3$, i.e.,

$|h_1^2 - h_2^2| \in H$ and $|\ell_1^2 - \ell_2^2| \in H$.

(ii) $|h^2 - \ell^2| = |1|$, i.e., $|h^2 - \ell^2| \in L$.

We may note that by division by three a nonnegative integer has the same remainder as the sum of its digits. Therefore, an immediate consequence of Lemma 2 is:

Lemma 3: Let Z be the set of all nonnegative integers and let $H = \{3z: z \in Z\}$, $L = Z \backslash H$. Then for any $h, h_1, h_2 \in H$ and $\ell, \ell_1, \ell_2 \in L$ we have

(i) $|S_{h_1}^2 - S_{h_2}^2| \in H$ and $|S_{\ell_1}^2 - S_{\ell_2}^2| \in H$;

(ii) $|S_h^2 - S_\ell^2| \in L$.

We now prove that the function $|S_x^t - S_y^t|$ is stable for $t = 2$. By Lemma 3 we see that e and b can play the same roles as shown in (G) if e represents the initial number which belongs to the set H and b represents the initial number which belongs to the set L. Thus we can find an integer $m \leq 4$ and four integers h_1, h_2, h_3, $h_4 \in H$ such that

$$f^m(x_1, x_2, x_3, x_4) = (h_1, h_2, h_3, h_4)$$

and S_{h_1}, S_{h_2}, S_{h_3}, $S_{h_4} \in H$. It follows that there exist four nonnegative integers h_5, h_6, h_7, h_8 such that

$$f^{m+1}(x_1, x_2, x_3, x_4) = (h_5, h_6, h_7, h_8)$$

and $S_{h_i} \equiv 0 \pmod 9$, $i = 5, 6, 7, 8$. On the other hand, if $\max\{h_5, h_6, h_7, h_8\}$ has four or more digits, then, after a finite number of steps (say d), we can find four nonnegative integers h_9, h_{10}, h_{11}, h_{12} such that

$$f^{m+d}(x_1, x_2, x_3, x_4) = (h_9, h_{10}, h_{11}, h_{12}),$$

where $S_{h_i} \equiv 0 \pmod 9$, $i = 9, 10, 11, 12$ and $\max\{h_9, h_{10}, h_{11}, h_{12}\} < 999$ (the proof is based on the same principle as shown in Steinhaus [2]). We know that $(9 + 9 + 9) = 27$. Therefore, $\max\{S_{h_9}, S_{h_{10}}, S_{h_{11}}, S_{h_{12}}\} < 27$. This indicates that the values of S_{h_i} ($i = 9, 10, 11, 12$) are either 0, 9, or 18. But we see that

$$18^2 - 0^2 = 324 \text{ and } 3 + 2 + 4 = 9;$$

$$18^2 - 9^2 = 234 \text{ and } 2 + 3 + 4 = 9;$$

$$18^2 - 18^2 = 0.$$

Thus, in the next step, we have

$$f^{m+d+1}(x_1, x_2, x_3, x_4) = (h_{13}, h_{14}, h_{15}, h_{16}),$$

where the values of S_{h_i} $(i = 13, 14, 15, 16)$ are either 0 or 9. It is easily verified that $f^c(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ for all $c \geq m + d + 5$. This shows that the function $|S_x^2 - S_y^2|$ is stable.

The function $|S_x^t - S_y^t|$ is not stable for $t \geq 3$. For instance, letting

$$g(x, y) = |S_x^3 - S_y^3|$$

and

$$x_1 = 21951,\ x_2 = 21609,\ x_3 = 0,\ x_4 = 324,$$

we have

$$\begin{array}{rrrrr} & (21951, & 21609, & 0, & 324) \\ f^1: & (\ \ 0, & 5832, & 729, & 5103) \\ f^2: & (\ 5832, & 0, & 5103, & 729) \\ f^3: & (\ 5832, & 729, & 5103, & 0) \end{array}$$

Since

$$f^3(21951, 21609, 0, 324) = f^1(21951, 21609, 0, 324) = (5832, 729, 5103, 0),$$

the process is periodic with period 2. The same result is obtained if we take (531441, 0, 426465, 104976) as the initial entries for $t = 4$.

The reader is welcome to consider the stability of the function $|T_x^t - T_y^t|$ in problem (2). About 500 quadruples of two-digit numbers (x_1, x_2, x_3, x_4) have been tested for $t = 2$ and $t = 3$. In each case, the functions $|T_x^2 - T_y^2|$ and $|T_x^3 - T_y^3|$ stabilized after 80 steps.

With respect to problem (3), it is not difficult to get an example to show that the function $T_x^2 + T_y^2 \pmod{32}$ is not stable. Letting

$$x_1 = 10,\ x_2 = 22,\ x_3 = 6,\ x_4 = 26,$$

we have

$$\begin{array}{rrrrr} & (10, & 22, & 6, & 26) \\ f^1: & (\ 9, & 12, & 12, & 9) \\ f^2: & (22, & 10, & 22, & 2) \\ f^3: & (\ 9, & 9, & 12, & 12) \end{array}$$

Thus, the process is periodic.

5. Ducci Processes in k-Dimensions

By analogy with Section 1, we now consider a function f whose domain and range are the set of k-tuples of nonnegative integers. Suppose that there is a function $g(x, y)$ whose domain is the set of pairs of nonnegative integers, whose range is the set of nonnegative integers, and that

$$f(x_1, x_2, \ldots, x_k) = (g(x_1, x_2), g(x_2, x_3), \ldots, g(x_k, x_1)).$$

Let $f^m(x_1, x_2, \ldots, x_k)$ be the mth iteration of f. Assume that entries of $f^m(x_1, x_2, \ldots, x_k)$ are bounded for all m (as before the bound depends on the initial choice of entries).

A Ducci process is a sequence of iterations of f. We call a function g stable if g generates a Ducci process such that for any choice of entries

$$f^{m+1}(x_1, x_2, \ldots, x_k) = f^m(x_1, x_2, \ldots, x_k) \text{ for some } m.$$

All of the Ducci processes in Sections 1-4 can be generalized to an arbitrary dimension k, where k is any integer greater than 2. We propose to examine only two such generalizations.

B. Freedman [3] proved that function $g(x, y) = |x - y|$ is stable if the number of members of the initial entries k is a power of 2.

We now show that the following functions are stable if and only if k is a power of 2.

(I) $g(x, y) = \overline{x + y} \pmod{2^n}$, $n = 1, 2, 3, \ldots$.

(II) $g(x, y) = \overline{x + y} \pmod{k}$, where k is an arbitrary positive integer.

Proof of (I): Let ${}^k f_i^m$ be the ith entry of the mth iteration of $f(x_1, x_2, \ldots, x_k)$. The subscript $i + j$ of x will always represent $\overline{i + j} \pmod{k}$.

Consider the function $g(x, y) = x + y$. We can show by mathematical induction that for any m,

$$ {}^k f_i^m = \sum_{j=0}^{m} \binom{m}{j} x_{i+j}. \tag{H}$$

In fact, (H) is true for $m = 1$, because

$$ {}^k f_i^1 = x_i + x_{i+1}.$$

Suppose (H) is true for m. Then

$$\begin{aligned} {}^k f_i^{m+1} &= {}^k f_i^1({}^k f_i^m) = \sum_{j=0}^{m} \binom{m}{j} x_{i+j} + \sum_{j=0}^{m} \binom{m}{j} x_{i+j+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \binom{m}{j} x_{i+j} + \sum_{j=0}^{m-1} \binom{m}{j} x_{i+j+1} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \binom{m}{j} x_{i+j} + \sum_{j=1}^{m} \binom{m}{j-1} x_{i+j} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m}{0} x_i + \sum_{j=1}^{m} \left[\binom{m}{j} + \binom{m}{j-1}\right] x_{i+j} + \binom{m}{m} x_{i+m+1} \\ &= \binom{m+1}{0} x_i + \sum_{j=1}^{m} \binom{m+1}{j} x_{i+j} + \binom{m+1}{m+1} x_{i+m+1} \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} x_{i+j}. \end{aligned}$$

Therefore, (H) is true for all m.

In particular, if k is a power of 2 ($k = 2^r$), then from (H) we have

$$\begin{aligned} {}^k f_i^k &= \sum_{j=0}^{2^r} \binom{2^r}{j} x_{i+j} = \binom{2^r}{0} x_i + \sum_{j=1}^{2^r-1} \binom{2^r}{j} x_{i+j} + \binom{2^r}{2^r} x_{i+2^r} \\ &= 2x_i + \sum_{j=1}^{2^r-1} \binom{2^r}{j} x_{i+j}. \end{aligned}$$

Adopting Freedman's technique, we see that $\binom{2^r}{j}$ is always even for $j = 1, 2, \ldots, 2^r - 1$. Hence

$$ {}^k f_i^k \equiv 0 \pmod{2},\ i = 1, 2, \ldots, k,\ k = 2^r$$

and

$${}^k f_i^{2k} \equiv 0 \pmod 4, \; i = 1, 2, \ldots, k, \; k = 2^r.$$

In general

$${}^k f_i^{tk} \equiv 0 \pmod{2^t}, \; i = 1, 2, \ldots, k, \; k = 2^r.$$

Thus, we conclude that for any n we have

$${}^k f_i^{nk} \equiv 0 \pmod{2^n}, \; i = 1, 2, \ldots, k, \; k = 2^r.$$

This means the function $g(x, y) = \overline{x + y} \pmod{2^n}$ is stable if k is a power of 2.

Proof of (II): The function $g(x, y) = \overline{x + y} \pmod k$ is stable if and only if $k = 2^r$ for any r. That this condition is sufficient follows from the previous proof. We now show that it is necessary.

We prove first that $g(x, y)$ is not stable if k is an odd prime p. Let the initial entries be $x_1, x_2, \ldots, x_p$, and

$$x_i = \begin{cases} 0 & \text{if } 0 < i < p \\ 1 & \text{if } i = p. \end{cases}$$

Then from (H) we have

$${}^p f_i^p = \begin{cases} \binom{p}{p - i} & \text{if } 0 < i < p \\ \binom{p}{p} + \binom{p}{0} & \text{if } i = p. \end{cases}$$

We know that $\binom{p}{p - i} = \binom{p}{i} \equiv 0 \pmod p$ for $0 < i < p$ when p is an odd prime. Hence,

$${}^p f_i^p = \begin{cases} 0 & \text{if } 0 < i < p \\ 2 & \text{if } i = p \end{cases}$$

and

$${}^p f_i^{pt} \equiv \begin{cases} 0 & \text{if } 0 < i < p \\ 2^t & \text{if } i = p \end{cases} \pmod p,$$

where t is a positive integer. Thus, by Fermat's theorem $2^{p-1} \equiv 1 \pmod p$, we obtain

$${}^p f_i^{p(p-1)} = \begin{cases} 0 & \text{if } 0 < i < p \\ 1 & \text{if } i = p. \end{cases}$$

Therefore, $g(x, y)$ is periodic.

Now let $k = ps$, where p is an odd prime and s is any integer greater than one. Let

$$x_i = \begin{cases} s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases} \quad *$$

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*For example, let $k = 6$; then $k = ps = 3 \times 2$. We set the initial entries as (0, 0, 2, 0, 0, 2). Thus we have

	(0, 0, 2, 0, 0, 2)
${}^6f^1$:	(0, 2, 2, 0, 2, 2)
${}^6f^2$:	(2, 4, 2, 2, 4, 2)
${}^6f^3$:	(0, 0, 4, 0, 0, 4)
${}^6f^4$:	(0, 4, 4, 0, 4, 4)
${}^6f^5$:	(4, 2, 4, 4, 2, 4)
${}^6f^6$:	(0, 0, 2, 0, 0, 2)

Then

$$ {}^k f_i^p = \begin{cases} 2s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise} \end{cases} $$

and

$$ {}^k f_i^{pt} \equiv \begin{cases} 2^t s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise} \end{cases} \quad (\text{mod } k), $$

where t is a positive integer. Hence

$$ {}^k f_i^{p(p-1)} = \begin{cases} s & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases} $$

Thus, function $g(x, y)$ is periodic and the proof is complete.

We leave it to the reader to examine generalizations of the Ducci processes presented in Sections 1-4.

References

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3. B. Freedman. "The Four Number Game." *Scripta Mathematica* 14 (1948):35-47.

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