CONSEQUENCES OF WATSON'S QUINTUPLE-PRODUCT IDENTITY

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1. Introduction

In this investigation, the leading role is played by the following identity:

$$(1)\qquad \prod_{n=1}^{\infty}(1-x^n)(1-ax^n)(1-a^{-1}x^{n-1})(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$

$$= \sum_{-\infty}^{\infty} x^{n(3n+1)/2}(a^{3n} - a^{-3n-1}),$$

which is valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. As presently expressed, identity (1) was first presented by Basil Gordon [2, p. 286]. However, as observed by M. V. Subbarao and M. Vidyasagar [5, p. 23], Gordon was anticipated some 32 years earlier by G. N. Watson [6, pp. 44-45], who stated and proved a fivefold-product identity easily shown to be equivalent to (1). We are here concerned about several applications of (1). Our first result is:

Theorem 1

For each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$,

$$(2)\qquad \prod_{n=1}^{\infty}(1-x^n)^2(1-ax^n)(1-a^{-1}x^n)(1-ax^{n-1})(1-a^{-1}x^{n-1})(1-a^2x^{2n-1})^2 \cdot (1-a^{-2}x^{2n-1})^2$$

$$= P(x)\sum_{-\infty}^{\infty} x^{3m^2}a^{6m} + Q(x)\sum_{0}^{\infty} x^{m(3m+1)}(a^{6m+1} + a^{-6m-1})$$

$$+ R(x)\sum_{0}^{\infty} x^{m(3m+2)}(a^{6m+2} + a^{-6m-2}) + S(x)\sum_{0}^{\infty} x^{3m(m+1)}(a^{6m+3} + a^{-6m-3})$$

$$+ T(x)\sum_{0}^{\infty} x^{m(3m+4)}(a^{6m+4} + a^{-6m-4}) + U(x)\sum_{0}^{\infty} x^{m(3m+5)}(a^{6m+5} + a^{-6m-5}),$$

where

$$P(x) = 2\sum_{-\infty}^{\infty} x^{k(3k+1)},\quad Q(x) = -\sum_{-\infty}^{\infty} x^{3k^2},\quad R(x) = -x\sum_{-\infty}^{\infty} x^{3k(k+1)},$$

$$S(x) = 2x\sum_{-\infty}^{\infty} x^{k(3k+2)}, \quad T(x) = -x^2\sum_{-\infty}^{\infty} x^{3k(k+1)}, \quad U(x) = -x^2\sum_{-\infty}^{\infty} x^{3k^2}.$$

The details of the proof are given in Section 2. As a corollary of Theorem 1, we then represent the decuple infinite product

$$\prod(1 - x^n)^6(1 - x^{2n-1})^4$$

by a double series in the single variable x. In Section 3 we shall need the following identity:

$$\prod_{n=1}^{\infty}(1 - x^n)^3(1 - x^{2n-1})^2 = \sum_{-\infty}^{\infty}(6n + 1)x^{n(3n+1)/2}, \tag{3}$$

shown by Gordon to be a fairly straightforward consequence of (1). On the strength of (3) and two other well-known identities, we then derive a recursive formula for the number-theoretic function $r_2(n)$, which for a given nonnegative integer n counts the number of representations of n as a sum of two squares.

2. Proof of Theorem 1

For given a, x let $G(a, x)$ be defined by:

$$G(a, x) = \prod_{n=1}^{\infty}(1 - ax^n)(1 - a^{-1}x^n)(1 - ax^{n-1})(1 - a^{-1}x^{n-1}) \cdot (1 - a^2x^{2n-1})^2(1 - a^{-2}x^{2n-1})^2.$$

Then, for each pair of positive real numbers A, X, with $X < 1$, $G(a, x)$ converges absolutely and uniformly on the set of all pairs a, x such that

$$A^{-1} \le |a| \le A \quad \text{and} \quad |x| \le X.$$

Hence, for a fixed choice of x, $|x| < 1$, $G(a, x)$ defines a unique function of a, which is analytic at all points of the finite complex plane except $a = 0$, where it has an essential singularity. Accordingly,

$$G(a, x) = C_0(x) + \sum_{n=1}^{\infty}[C_n(x)a^n + C_{-n}(x)a^{-n}],$$

where the coefficients $C_n(x)$, $C_{-n}(x)$ are uniquely determined by the chosen x.

Now, $G(a, x) = G(a^{-1}, x)$, whence $C_n(x) = C_{-n}(x)$, for each positive integer n. Hence,

$$G(a, x) = C_0(x) + \sum_{n=1}^{\infty}C_n(x)(a^n + a^{-n}). \tag{4}$$

An easy calculation then establishes the following identity:

$$G(ax, x) = a^{-6}x^{-3}G(a, x).$$

With the help of (4) we expand both sides of this identity in powers of a, and subsequently equate coefficients of like powers to obtain the following recurrence:

$$C_n(x) = C_{n-6}(x)x^{n-3}.$$

The coefficients $C_0(x)$, $C_1(x)$, $C_2(x)$, $C_3(x)$, $C_4(x)$, $C_5(x)$ are here undetermined, but for all $n > 5$, we distinguish six cases,

$$\text{(i)}\quad n = 6m, \qquad \text{(ii)}\quad n = 6m+1, \qquad \text{(iii)}\quad n = 6m+2,$$

$$\text{(iv)}\quad n = 6m+3, \qquad \text{(v)}\quad n = 6m+4, \qquad \text{(vi)}\quad n = 6m+5,$$

$m \geq 0$, and iterate the recurrence to obtain:

$$C_{6m}(x) = x^{3m^2}C_0(x),\ C_{6m+1}(x) = x^{m(3m+1)}C_1(x),\ C_{6m+2}(x) = x^{m(3m+2)}C_2(x),$$

$$C_{6m+3}(x) = x^{3m(m+1)}C_3(x),\ C_{6m+4}(x) = x^{m(3m+4)}C_4(x),\ C_{6m+5}(x) = x^{m(3m+5)}C_5(x).$$

Hence,

$$\begin{aligned}(5)\quad G(a, x) = {} & C_0(x)\sum_{-\infty}^{\infty} x^{3m^2}a^{6m} + C_1(x)\sum_{0}^{\infty} x^{m(3m+1)}(a^{6m+1} + a^{-6m-1}) \\ & + C_2(x)\sum_{0}^{\infty} x^{m(3m+2)}(a^{6m+2} + a^{-6m-2}) \\ & + C_3(x)\sum_{0}^{\infty} x^{3m(m+1)}(a^{6m+3} + a^{-6m-3}) \\ & + C_4(x)\sum_{0}^{\infty} x^{m(3m+4)}(a^{6m+4} + a^{-6m-4}) \\ & + C_5(x)\sum_{0}^{\infty} x^{m(3m+5)}(a^{6m+5} + a^{-6m-5}).\end{aligned}$$

To evaluate C_0, C_1, C_2, C_3, C_4, and C_5, we multiply identity (1) and the identity which results from (1) under the substitution $a \to a^{-1}$ to get

$$\begin{aligned}\prod_{n=1}^{\infty}(1 - x^n)^2 G(a, x) = {} & P(x)a^0 + Q(x)(a + a^{-1}) + R(x)(a^2 + a^{-2}) \\ & + S(x)(a^3 + a^{-3}) + T(x)(a^4 + a^{-4}) \\ & + U(x)(a^5 + a^{-5}) + \text{a series in } a^n, a^{-n}, n > 5.\end{aligned}$$

Between identity (5) and the foregoing identity, we eliminate the product $G(a, x)$ and, thereafter, equate coefficients of a^0, $a + a^{-1}$, ..., $a^5 + a^{-5}$ to get

$$C_0 = P(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_1 = Q(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_2 = R(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},$$

$$C_3 = S(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_4 = T(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2},\ C_5 = U(x)\prod_{n=1}^{\infty}(1 - x^n)^{-2}.$$

Substituting these values of C_i ($i = 0, 1, \ldots, 5$) into (5) we thus prove our theorem.

Corollary

For each complex number x such that $|x| < 1$,

$$\begin{aligned}(6)\qquad \prod_{n=1}^{\infty}(1 - x^n)^6(1 - x^{2n-1})^4 = &-\sum_{-\infty}^{\infty} x^{k(3k+1)}\sum_{-\infty}^{\infty}(6m)^2x^{3m^2}\\ &+\sum_{-\infty}^{\infty} x^{3k^2}\sum_{-\infty}^{\infty}(6m + 1)^2x^{m(3m+1)}\\ &+x\sum_{-\infty}^{\infty} x^{3k(k+1)}\sum_{-\infty}^{\infty}(6m + 2)^2x^{m(3m+2)}\\ &-x\sum_{-\infty}^{\infty} x^{k(3k+2)}\sum_{-\infty}^{\infty}(6m + 3)^2x^{3m(m+1)}.\end{aligned}$$

Proof: For given a, x, let $F(a, x)$ be defined by

$$(1 - a)(1 - a^{-1})F(a, x) = \prod_{n=1}^{\infty}(1 - x^n)^2G(a, x),$$

which is the left side of (2). Now, put $a = e^{2it}$, and for brevity

$$f(t) = F(e^{2it}, x).$$

Identity (2) is hereby transformed into a new identity, the left side of which is $4f(t)\sin^2 t$. Hence, we multiply both sides of this new identity by 4^{-1} to get

$$\begin{aligned}f(t)\sin^2 t = &\frac{P(x)}{4}\left[1 + 2\sum_{n=1}^{\infty}x^{3m^2}\cos(12mt)\right] + \frac{Q(x)}{2}\sum_{0}^{\infty}x^{m(3m+1)}\cos(12m + 2)t\\ &+\frac{R(x)}{2}\sum_{0}^{\infty}x^{m(3m+2)}\cos(12m + 4)t + \frac{S(x)}{2}\sum_{0}^{\infty}x^{3m(m+1)}\cos(12m + 6)t\\ &+\frac{T(x)}{2}\sum_{0}^{\infty}x^{m(3m+4)}\cos(12m + 8)t + \frac{U(x)}{2}\sum_{0}^{\infty}x^{m(3m+5)}\cos(12m + 10)t.\end{aligned}$$

We now differentiate the foregoing identity twice with respect to t to get

$$2f(t)\cos^2 t + 2\sin t\, D_t[f(t)\cos t] + D_t[f'(t)\sin^2 t]$$

$$= -2P(x)\sum_{1}^{\infty}x^{3m^2}(6m)^2\cos(12mt) - 2Q(x)\sum_{0}^{\infty}x^{m(3m+1)}(6m + 1)^2\cos(12m + 2)t$$

$$- 2R(x)\sum_0^\infty x^{m(3m+2)}(6m+2)^2\cos(12m+4)t$$

$$- 2S(x)\sum_0^\infty x^{3m(m+1)}(6m+3)^2\cos(12m+6)t$$

$$- 2T(x)\sum_0^\infty x^{m(3m+4)}(6m+4)^2\cos(12m+8)t$$

$$- 2U(x)\sum_0^\infty x^{m(3m+5)}(6m+5)^2\cos(12m+10)t.$$

In the foregoing we first put $t = 0$ and cancel a factor of 2 from both sides of the resulting identity. Of course, $f(0)$ is the left side of (6). To get the right side, we then combine the 2nd and 6th, and the 3rd and 5th sums on the right side of the last-mentioned identity, while effecting some fairly obvious transformations along the way.

3. Recurrences for $r_2(n)$

In order to carry out our present assignment, we also need the following well-known identities:

$$\prod_{n=1}^\infty (1 - x^n) = \sum_{-\infty}^\infty (-1)^n x^{n(3n+1)/2}, \tag{7}$$

$$\prod_{n=1}^\infty (1 - x^n)(1 - x^{2n-1}) = \sum_{-\infty}^\infty (-x)^{n^2}. \tag{8}$$

(7) is a famous result due to Euler, and both identities are easy consequences of the celebrated Gauss-Jacobi triple-product identity [3, pp. 282-284].

For convenience, put $r(n) = r_2(n)$.

Theorem 2

For each nonnegative integer n,

$$r(n) + \sum_{j=1}[(-1)^{j(3j-1)/2}r(n - (3j+1)/2) + (-1)^{j(j-1)/2}r(n - (3j-1)/2)] \tag{9}$$

$$= \begin{cases} (-1)^n[6(\pm m) + 1], & \text{if } n = m(3m \pm 1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

where summation extends as far as the arguments of r remain nonnegative.

Proof: First of all, we recall that the generating function of $r(n)$ is given by:

$$\left(\sum_{-\infty}^{\infty} x^{n^2}\right)^2 = \sum_{n=0}^{\infty} r(n)x^n.$$

We now realize that (3) is equivalent to

$$\prod_1^\infty (1 - x^n) \prod_1^\infty (1 - x^n)^2(1 - x^{2n-1})^2 = \sum_{-\infty}^{\infty} (6n + 1)x^{n(3n+1)/2},$$

whence [owing to (7) and (8)]

$$\sum_{-\infty}^{\infty} (-1)^n x^{n(3n+1)/2} \sum_0^\infty r(n)(-x)^n = \sum_{-\infty}^{\infty} (6n + 1)x^{n(3n+1)/2}.$$

Expanding the left side of the foregoing identity and thereafter equating coefficients of like powers of x, we obtain the desired conclusion.

Remarks

It is of interest to compare the recursive determination (9) of the arithmetical function r with similar ones for the partition function p and the sum-of-divisors function σ. Accordingly, let us briefly recall that for a given positive integer n, $p(n)$ denotes the number of unrestricted partitions of n, while $\sigma(n)$ denotes the sum of the positive divisors of n; conventionally, $p(0) = 1$. From his identity, Euler derived the following recursive formulas for p and σ.

$$(10) \qquad p(n) + \sum_{j=1} (-1)^j[p(n - j(3j + 1)/2) + p(n - j(3j - 1)/2)] = 0,$$

where $n > 0$ and summation extends as far as the arguments of p remain nonnegative.

$$(11) \qquad \sigma(n) + \sum_{j=1} (-1)^j[\sigma(n - j(3j + 1)/2) + \sigma(n - j(3j - 1)/2)] = \begin{cases} (-1)^{m+1}n, & \text{if } n = m(3m \pm 1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

where $n > 0$ and summation extends as far as the arguments of σ remain positive.

For proofs of (10) and (11), see [4, pp. 235-237].

Thus, for these three important arithmetical functions r, p, and σ, we have pentagonal-number recursive formulas for each of them. And for each of them one needs about $2\sqrt{(2/3)n}$ of the earlier values to compute a given value for large n.

In [1] the author has also derived the following triangular-number recursive formula for r:

$$\sum_{j=0} (-1)^{j(j+1)/2} r(n - j(j+1)/2) = \begin{cases} (-1)^{m(m+3)/2}(2m+1), & \text{if } n = m(m+1)/2, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

where $n \geq 0$ and summation extends as far as the arguments of r remain nonnegative.

We now observe that recursive formula (12) is more efficient than (9). For with (12) one needs about $\sqrt{2n}$ of the earlier values in order to compute $r(n)$ for large n.

References

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