THEVENIN EQUIVALENTS OF LADDER NETWORKS

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An electrical network of considerable importance in applications is known as the *ladder network*. A common form of this circuit consists of resistive elements connected together as shown in Figure 1. It is often used as an attenuator to reduce the applied input voltage to various other values which are made available to subsequent loads through the m taps shown in the same figure.

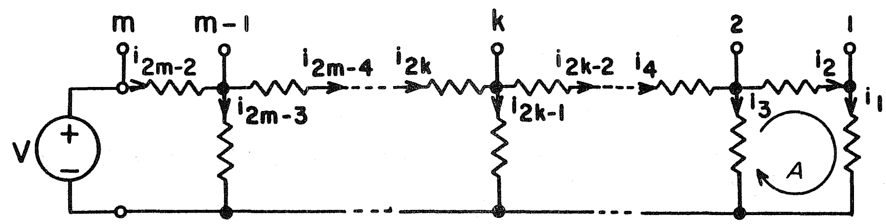

Figure 1

A basic result of elementary circuit analysis is that any network of linear resistors and sources may be replaced by an equivalent circuit consisting of an ideal voltage source and a single series resistor. This configuration is known as the *Thevenin equivalent of the original network*. It is often desirable to find the Thevenin equivalent voltage and resistance of a ladder network as perceived by a load connected to one of its taps. The case in which all the resistors in the ladder network have identical values is of particular interest since the expressions for the Thevenin equivalents involve the Fibonacci sequence.

The derivation of these expressions requires the use of three basic rules of circuit analysis and one observation. The three rules are known as *Kirchoff's voltage law*, *Kirchoff's current law*, and *Ohm's law* (for a full discussion of these, see [1]). The observation is that in a ladder network such as that shown in Figure 1, the current in the jth resistor is related to that in the rightmost resistor by:

$$i_j = F_j i_1,\ j = 1, 2, \ldots, 2m - 2.$$

To derive an expression for the Thevenin equivalent voltage at the kth tap, with a given input voltage v, one must find the voltage appearing at that tap with the tap open-circuited. Under these conditions:

$$v_k = Ri_{2k-1} = F_{2k-1}Ri_1,$$

where R is the common value of all the resistors. Applying Kirchoff's voltage law to the leftmost loop yields:

$$v = (i_{2m-2} + i_{2m-3})R = F_{2m-1}Ri_1$$

or

$$i_1 = \frac{v}{RF_{2m-1}}.$$

Hence,

$$v_k = v\left(\frac{F_{2k-1}}{F_{2m-1}}\right), \text{ for } k = 1, 2, \ldots, m.$$

Derivation of an expression for the Thevenin equivalent resistance at the kth tap requires the application of a principle of circuit analysis which says that the Thevenin equivalent resistance of a network may be found by evaluating the effective resistance of the network after all independent sources have been set equal to zero. In this case, the mth tap must be shorted to ground to eliminate the voltage source supplying the input voltage v. To determine the effective resistance once this is done, a current source of unit value may be applied at the kth tap. If the voltage at the kth tap can be determined, the Thevenin equivalent resistance may then be found from Ohm's law.

Applying the unit current source to the kth tap, as shown in Figure 2, Kirchoff's current law at the kth tap becomes:

$$i_c + i_d + i_e = 1.$$

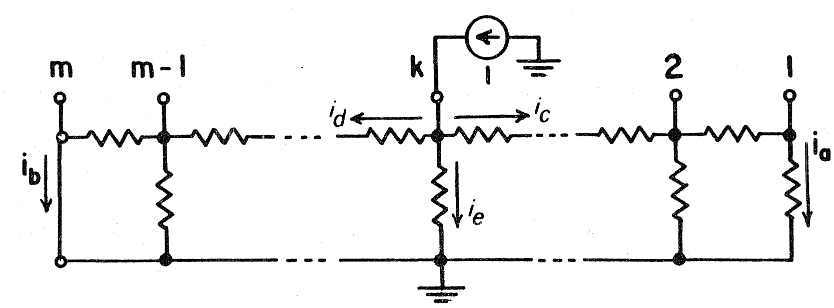

Figure 2

What is the relationship between i_c, i_d, and i_e? We previously cited the observation that, in the circuit of Figure 1, the current in the jth resistor was related to that in the rightmost resistor by:

$$i_j = F_j i_1, \; j = 1, 2, \ldots, 2m - 2.$$

This was obtained from examining the results of applying Kirchoff's voltage law to loop A in Figure 1, then applying Kirchoff's current law to node 2, and so forth, each time relating the currents and voltages back to i_1. Since these relationships depend on the way the resistors are connected, they are still valid for resistors to the right of the kth tap in Figure 2. Hence, we obtain

$$i_c = F_{2k-2} i_a.$$

If we start at the left end and work rightward, alternately writing loop and node equations and relating the voltages and currents back to i_b, we can similarly obtain

$$i_d = F_{2(m-k)-1} i_b.$$

Working again from the right end, we find

$$i_e = F_{2k-1} i_a.$$

Working from the left end, we find

$$i_e = F_{2(m-k)} i_b.$$

The current i_e can be eliminated to give a relationship between i_a and i_b, namely

$$F_{2k-1} i_a = F_{2(m-k)} i_b.$$

Now, we may replace i_c, i_d, and i_e in Kirchoff's current law at the kth tap. We obtain:

$$F_{2k-2} i_a + F_{2(m-k)-1} i_b + F_{2k-1} i_a = 1.$$

Eliminating i_b gives us:

(1) $$i_a \left(F_{2k-2} + \frac{[F_{2(m-k)-1}][F_{2k-1}]}{F_{2(m-k)}} + F_{2k-1} \right) = 1.$$

Now, the voltage at the kth tap is $v_k = Ri_e$. The Thevenin equivalent resistance is then

(2) $$R_k = v_k/1 = Ri_e = R[F_{2k-1} i_a].$$

Solving for i_a from Eq. (1) above and substituting it into the expression for R_k (2), we obtain:

(3) $$R_k = R\left(\frac{F_{2k-1}}{F_{2k} + \dfrac{[F_{2(m-k)-1}][F_{2k-1}]}{F_{2(m-k)}}} \right), \; k = 1, 2, \ldots, m.$$

The Fibonacci sequence is thus seen to insinuate itself into the expression for ladder network Thevenin equivalents, chiefly as a result of the manner in which currents are related in the network. These results may be of some practical value in affording a simple means of analyzing a particular ladder network. If nothing else, they provide an interesting example of the occurrence of the Fibonacci sequence in an applied situation.

References

1. Donald A. Calahan, Alan B. Macnee, and E. Lawrence McMahon. *Introduction to Modern Circuit Analyais*. New York: Holt, Rinehart and Winston, 1974.
2. S. L. Basin. "The Fibonacci Sequence as it Appears in Nature." *The Fibonacci Quarterly* 1, no. 1 (Feb. 1963):53-66.
3. S. L. Basin. "The Appearance of the Fibonacci Numbers and the Q Matrix in Electrical Network Theory." *Math. Mag.*, March-April, 1963, pp. 84-97.

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