# A TRINOMIAL DISCRIMINANT FORMULA

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The expression $b^2 - 4ac$ is well known to algebra students as the discriminant of the quadratic $ax^2 + bx + c$, with $a \neq 0$. However, how many students are aware of the existence of discriminant formulas for higher-degree polynomials? The purpose of this paper is to develop such a formula for the trinomial

$$ax^n + bx^k + c, \tag{1}$$

with $n > k > 0$ and $a \neq 0$. The formula has appeared in the literature in various forms ([1, p. 130], [2], [3], [4], [5, p. 41], and [6]). It can be written as

$$\Delta_{n,k} = (-1)^{\frac{1}{2}n(n-1)}a^{n-k-1}c^{k-1}(n^N a^K c^{N-K} + (-1)^{N-1}(n - k)^{N-K}k^K b^N)^d, \tag{2}$$

where $d$ is the greatest common divisor of $n$ and $k$ and $N$ and $K$ are given by $n = Nd$ and $k = Kd$. Notice that the case $n = 2$ and $k = 1$ gives the quadratic discriminant

$$\Delta_{2,1} = b^2 - 4ac.$$

In this paper we derive (2) by standard algebraic techniques that involve some elementary calculus and roots of unity. As a generalization of the quadratic case, the trinomial discriminant formula can provide an interesting enrichment topic for advanced-level algebra students.

To appreciate what is involved in deriving (2), consider the usual definition of the discriminant $D_n$ of the general $n$th-degree polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n. \tag{3}$$

Van der Waerden [7, p. 101], for example, defines $D_n$ as

$$D_n = a_0^{2n-2}\prod_{i<j}(\alpha_i - \alpha_j)^2, \tag{4}$$

where the $\alpha$'s are the roots of $f(x)$.

As examples, let us compute $D_n$ for $n = 2$ and $n = 3$. In these cases, (3) is more commonly written as $f(x) = ax^2 + bx + c$ and $f(x) = ax^3 + bx^2 + cx + d$, respectively. Using (4) together with the well-known expressions that relate the coefficients of each polynomial to the elementary symmetric functions of their roots, we get

$$D_2 = b^2 - 4ac$$

and

$$D_3 = b^2c^2 - 27a^2d^2 - 4b^3d - 4ac^3 + 18abcd.$$

We note that for $n \geqslant 3$, $D_n$ becomes more difficult to compute directly from the roots of $f(x)$.

There are other expressions for $D_n$ that involve the derivative $f'$ of (3). A straightforward manipulation of the product (4), for example, gives:

$$D_n = (-1)^{\frac{1}{2}n(n-1)}a_0^{n-2}\prod_{i=1}^{n} f'(\alpha_i). \tag{5}$$

Still another expression for $D_n$ is the one we will use to derive (2), namely:

$$D_n = (-1)^{\frac{1}{2}n(n-1)}a_0^{n-1}n^n\prod_{j=1}^{n-1} f(\beta_j), \tag{6}$$

where the $\beta$'s are the roots of $f'(x)$. It is not hard to compute the discriminant of (1) from (6) because the derivative of a trinomial is a binomial whose roots are easy to find.

The expression (6) is obtained by considering the double product

$$(a_0n)^n \prod_{i=1}^{n}\prod_{j=1}^{n-1}(\alpha_i - \beta_j),$$

where the $\alpha_i$'s and the $\beta_j$'s are the roots of $f(x)$ and $f'(x)$, respectively. By rearranging this double product, as described in [7], it is easy to show that it is equal to each of the following single products, which are hence equal to each other:

$$a_0n^n\prod_{j=1}^{n-1} f(\beta_j) = \prod_{i=1}^{n} f'(\alpha_i). \tag{7}$$

A comparison of (7) with (5) then gives (6).

We now derive the discriminant formula. We first obtain the formula for $f(x) = ax^n - bx^k + c$ and then replace $b$ by $-b$. Write

$$f(x) = ax^n - bx^k + c = c - (b - ax^{n-k})x^k \tag{8}$$

and

$$f'(x) = nax^{n-1} - kbx^{k-1} = x^{k-1}(nax^{n-k} - kb).$$

Clearly, the roots of the binomial $f'(x)$ are $(k - 1)$ zeros and the solutions of $x^{n-k} = kb/na$. Therefore, by (8),

$$\prod_{j=1}^{n-1} f(\beta_j) = c^{k-1}\prod_{\zeta}\left(c - (b - a(\zeta\beta)^{n-k})(\zeta\beta)^k\right),$$

where $\zeta$ runs through all of the $(n - k)$th roots of unity and $\beta^{n-k} = kb/na$. For further information about roots of unity, see [7, Sec. 36]. Simplifying, we have

$$\prod_{j=1}^{n-1} f(\beta_j) = c^{k-1} \prod_{\zeta}\left(c - \left(\frac{n-k}{n}\right)b\ \beta^k \zeta^k\right).$$

Now, as $\zeta$ runs through the $(n - k)$th roots of unity, $\zeta^k$ runs $d$ times through the $(N - K)$th roots of unity. Therefore, after further simplification with roots of unity, we get

$$\prod_{j=1}^{n-1} f(\beta_j) = c^{k-1}\left(c^{N-K} - (n - k)^{N-K} k^K n^{-N} a^{-K} b^N\right)^d.$$

Here we are using the fact that, if $\omega$ is a primitive $m$th root of unity, then

$$u^m - v^m = \prod_{i=0}^{m-1} (u - v\omega^i).$$

Using (6) and substituting $-b$ for $b$, we obtain the desired formula given in (2).

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