◆◇◆◇◆

ONE-FREE ZECKENDORF SUMS

CLARK KIMBERLING
University of Evansville, Evansville, IN 47702
(Submitted December 1981)

The main theorem about representations of positive integers as sums of Fibonacci numbers, widely known as Zeckendorf's Theorem even before it was published [8], states that every positive integer is a sum of nonconsecutive Fibonacci numbers and that this representation is unique. Examples of such sums follow:

$$11 = 3 + 8,\ 12 = 1 + 3 + 8,\ 13 = 13,\ 70 = 2 + 13 + 55.$$

Zeckendorf's Theorem implies that the sums of distinct Fibonacci numbers form the sequence of all positive integers. It is the purpose of this note to prove that the sums of distinct terms of the *truncated* Fibonacci sequence (2, 3, 5, 8, ...) form the sequence

$$[(1 + \sqrt{5})n/2] - 1,\ n = 2, 3, 4, \ldots .$$

We shall use the usual notation for Fibonacci numbers, the greatest integer function, and fractional parts:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \text{ for } n = 1, 2, 3, \ldots;$$

$$[x] = \text{the greatest integer} \leqslant x; \text{ and}$$

$$\{x\} = x - [x].$$

A well-known connection between the number $\alpha = (1 + \sqrt{5})/2$ and F_n, to be used in the sequel, is that $[\alpha F_n] = F_{n+1}$ if n is odd and $= F_{n+1} - 1$ if n is even.

Lemma 1

Let n and c be positive integers satisfying $n \geqslant 2$ and $1 \leqslant c \leqslant F_n$. Let $S = \{\alpha c\} + \{\alpha F_n\}$. Then $S < 1$ for odd n and $S > 1$ for even n.

Proof: It is well known (e.g. [6, p. 101]) that

$$\frac{1}{F_{n+2}F_{n+4}} < \left|\alpha - \frac{F_{n+2}}{F_{n+1}}\right| < \frac{1}{F_{n+2}F_{n+3}}.$$

Shifting the index and multiplying by F_n gives

$$F_n/F_{n+1}F_{n+3} < \{\alpha F_n\} < F_n/F_{n+1}F_{n+2} \text{ for odd } n, \tag{1}$$

and

$$1 - F_n/F_{n+1}F_{n+2} < \{\alpha F_n\} < 1 - F_n/F_{n+1}F_{n+3} \text{ for even } n. \tag{2}$$

Now F_n/F_{n-1} is a best approximation of α, which means that

$$|\alpha F_{n-1} - F_n| \leqslant |\alpha e - d| \tag{3}$$

for all integers d and e satisfying $0 < e \leqslant F_n$.

Case 1. Suppose n is odd. Then (3) with $d = [\alpha c + 1]$ implies

$$F_n - \alpha F_{n-1} \leqslant [\alpha c + 1] - \alpha c,$$

so that $1 - \{\alpha F_{n-1}\} \leqslant 1 - \{\alpha c\}$, or equivalently, $\{\alpha c\} \leqslant \{\alpha F_{n-1}\}$. Thus

$$\begin{aligned} S &\leqslant \{\alpha F_{n-1}\} + \{\alpha F_n\} \\ &< 1 - F_{n-1}/F_n F_{n+2} + F_n/F_{n+1}F_{n+2} \text{ by (1) and (2)} \\ &= 1 - 1/F_n F_{n+1}F_{n+2} \\ &< 1. \end{aligned}$$

Case 2. Suppose n is even. Then (3) implies $\{\alpha F_{n-1}\} \leqslant 1 - \{\alpha c\}$, so that

$$\begin{aligned} S &\geqslant 1 - \{\alpha F_{n-1}\} + \{\alpha F_n\} \\ &> 1 - F_{n-1}/F_n F_{n+1} + 1 - F_n/F_{n+1}F_{n+2} \\ &= 2 - F_{n+1}/F_n F_{n+2} \\ &> 1. \end{aligned}$$

Lemma 2

Let n and c be positive integers satisfying $n \geqslant 2$ and $1 \leqslant c \leqslant F_n$. Then

$$[(\alpha + 1)(c + F_n) - 1] = [(\alpha + 1)c - 1] + F_{n+2}.$$

Proof: If n is odd and $\geqslant 3$, then

$$\begin{aligned} [(\alpha + 1)(c + F_n)] &= [(\alpha + 1)c] + [(\alpha + 1)F_n] \text{ by Lemma 1} \\ &= [(\alpha + 1)c] + F_n + [\alpha F_n] \\ &= [(\alpha + 1)c] + F_n + F_{n+1} \\ &= [(\alpha + 1)c] + F_{n+2}. \end{aligned}$$

If n is even, then

$$\begin{aligned}[(\alpha + 1)(c + F_n)] &= [(\alpha + 1)c] + [(\alpha + 1)F_n] + 1\\ &= [(\alpha + 1)c] + F_n + [\alpha F_n] + 1\\ &= [(\alpha + 1)c] + F_n + F_{n+1}\\ &= [(\alpha + 1)c] + F_{n+2}.\end{aligned}$$

Lemma 3

If M is a positive integer whose Zeckendorf sum uses 1, then there exists a positive integer C such that $M = [(\alpha + 1)C - 1]$. Explicitly, if

$$M = 1 + F_{n_1} + F_{n_2} + \cdots + F_{n_k} \text{ where } 4 \leqslant n_i \leqslant n_{i+2} - 1, \quad i = 1, 2, \ldots, k - 2, \tag{4}$$

then

$$C = 1 + F_{n_1 - 2} + F_{n_2 - 2} + \cdots + F_{n_k - 2}.$$

Proof: As a first step, $1 = [\alpha]$. Now, suppose $M > 1$ has Zeckendorf sum (4) and, as an induction hypothesis, that if m is any positive integer $< M$, then, in terms of its Zeckendorf sum

$$m = 1 + F_{u_1} + F_{u_2} + \cdots + F_{u_v},$$

we have $m = [(\alpha + 1)c - 1]$, where

$$c = 1 + F_{u_1 - 2} + F_{u_2 - 2} + \cdots + F_{u_v - 2}.$$

Let $c' = 1 + F_{n_1 - 2} + F_{n_2 - 2} + \cdots + F_{n_{k-1} - 2}$. Then

$$c' \leqslant \sum_{j=2}^{n_{k-1}-2} F_j = -2 + F_{n_{k-1}} < F_{n_k - 2}.$$

Lemma 2 therefore applies:

$$[(\alpha + 1)(c' + F_{n_k - 2}) - 1] = [(\alpha + 1)c' - 1] + F_{n_k},$$

and by the induction hypothesis, this equals

$$(1 + F_{n_1} + F_{n_2} + \cdots + F_{n_{k-1}}) + F_{n_k},$$

so that Lemma 3 is proved.

Lemma 4

The set of all positive integers C of the form

$$1 + F_{n_1-2} + F_{n_2-2} + \cdots + F_{n_k-2},\ n \text{ as in (4)}, \tag{5}$$

together with 1, is the set of all positive integers.

Proof: Let C be any positive integer > 1 and let $C - 1$ have Zeckendorf sum

$$F_{u_1} + F_{u_2} + \cdots + F_{u_j}.$$

(If $F_{u_1} = 1$, it is understood that $u_1 = 2$.) Then C equals the sum (5) with $j = k$ and $n_i = u_i + 2$ for $i = 1, 2, \ldots, k$.

Theorem

The sums of distinct terms of the truncated Fibonacci sequence

$$(2, 3, 5, 8, \ldots)$$

form the sequence

$$[\alpha n - 1],\ n = 2, 3, 4, \ldots .$$

Proof: By Lemmas 3 and 4, the set of positive integers that are *not* such sums forms the sequence

$$[(\alpha + 1)n - 1],\ n = 1, 2, 3, \ldots .$$

Applying Beatty's method (based on a famous problem published in [1]) to the sequence $[(\alpha + 1)n]$, we conclude that the complement of this sequence is $[\alpha n]$. The complement of $[(\alpha + 1)n - 1]$ in the *positive* integers is therefore $[\alpha n - 1]$, $n = 2, 3, 4, \ldots$.

Remarks:

1. The first 360 terms of the sequence $[\alpha n - 1]$, i.e., the first 360 positive integers whose Zeckendorf sums do require 1, are listed in [2, pp. 62-64].

2. Fraenkel, Levitt, & Shimshoni [4] observe in their Corollary 1.3 that a certain property relating to Zeckendorf-type sums holds if and only if α has the form

$$\frac{1}{2}(2 - a + \sqrt{a^2 + 4})$$

for some positive integer a. When $a = 2$, we have $\alpha = \sqrt{2}$, and the sequence analogous to 1, 2, 3, 5, 8, 13, ... is 1, 3, 7, 17, 41, 99, ... The first few numbers expressible as Zeckendorf-type sums of the truncated sequence 3, 7, 17, 41, 99, ... (see [4, p. 337, item (i), for a precise definition of Zeckendorf-type sums in this setting) are 3, 6, 7, 10, 13, 14. Sequences of the form $[\gamma n + \delta]$ cannot yield 3, 6, 7, consecutively. Therefore, Corollary 1.3 of [4] offers no immediate generalization of the theorem on sums from the truncated Fibonacci sequence. Does any nontrivial generalization exist?

3. The interested reader should consult Fraenkel, Levitt, & Shimshoni [4]. Their Theorem 1 states that for $\alpha = (1 + \sqrt{5})/2$, the numbers $[n\alpha]$ are "even" P-system numbers (= Zeckendorf sums, although they are not so named in [4]) and the numbers $[n\beta]$ are "odd." The one-free Zeckendorf sums discussed in this present work are $[n\alpha - 1]$, some of which are even and some of which are odd in the sense of [4]. Being one-free is equivalent to ending in zero in [4]; however, the attention in [4] is on the number of terminal zeros—whether that number is even or odd, and no criterion is given in [4] for whether the terminal digit is zero.

REFERENCES

1. S. Beatty. Problem 3177. *American Math. Monthly* 33 (1926):159; 34 (1927):159.

2. Bro. Alfred Brousseau. *Fibonacci and Related Number Theoretic Tables.* Santa Clara, Calif.: The Fibonacci Association, 1972.

3. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers." *J. London Math. Soc.* 35 (1960):143-160.

4. A. Fraenkel, J. Levitt, & M. Shimshoni. "Characterization of the Set of Values $f(n) = [n\alpha]$, $n = 1, 2, \ldots$". *Discrete Math.* 2 (1972):335-345.

5. C. G. Lekkerkerker. "Voorstelling van Fibonacci." *Simon Stevin* 29 (1951-52):190-195.

6. J. Roberts. *Elementary Number Theory.* Cambridge: MIT Press, 1977.

7. K. Stolarsky. "Beatty Sequences, Continued Fractions, and Certain Shift Operators." *Canadian Math. Bull.* 19 (1976):473-482.

8. E. Zeckendorf. "Représentation des nombres naturels par une soome de nombres de Fibonacci ou de nombres de Lucas." *Bull. Soc. Royale Sci. Liège* 41 (1972):179-182.

◆◇◆◇◆