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ON FIBONACCI AND LUCAS NUMBERS WHICH ARE SUMS OF PRECISELY FOUR SQUARES

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INTRODUCTION

A well-known theorem of Lagrange states that every positive integer is a sum of four squares [4, p. 302]. In this article we determine which Fibonacci and Lucas numbers are sums of not fewer than four positive squares. The nth Fibonacci and Lucas numbers are denoted $F(n), L(n)$, respectively, in order to avoid the need for subscripts that carry exponents.

PRELIMINARIES

(1) $m \neq a^2 + b^2 + c^2$ iff $m = 4^j k$, with $j \geqslant 0$ and $k \equiv 7 \pmod 8$
(2) $F(2n) = F(n)L(n)$
(3) $L(2n) = L(n)^2 - 2(-1)^n$
(4) $F(m + n) = F(m)F(n - 1) + F(m + 1)F(n)$
(5) $F(12n \pm 1) \equiv 1 \pmod 8$
(6) $F(n) \equiv 7 \pmod 8$ iff $n \equiv 10 \pmod{12}$
(7) $F(n) \equiv 0 \pmod 4$ implies $F(n) \equiv 0 \pmod 8$
(8) $L(n) \not\equiv 0 \pmod 8$
(9) $L(n) \equiv 7 \pmod 8$ iff $n \equiv 4, 8,$ or $11 \pmod{12}$
(10) $L(n) \equiv 28 \pmod{32}$ iff $n \equiv 21 \pmod{24}$
(11) $L(12n) \equiv 2 \pmod{32}$
(12) If $j \geqslant 2$, then $4^j \mid F(n)$ iff $n = 3(4^{j-1})m$, with $(6, m) = 1$.

Remarks: (1) is stated on p. 311 of [4]. (2) and (3) are 12b, d, and e on p. 101 of [1]. (4) is (1) on p. 289 of [2]. (5), (6), and (7) are established by observing the periodic residues of the Fibonacci sequence (mod 8), namely: 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, etc. (8) and (9) are established by observing the periodic residues of the Lucas sequence (mod 8), namely: 2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1, etc. (10) and (11) are established by observing the periodic residues of the Lucas sequence (mod 32), namely: 2, 1, 3, 4, 7, 11, 18, 29, 15, 12, 27, 7, 2, 9, 11, 20, 31, 19, 18, 5, 23, 28, 19, 15, 2, 17, 19, 4, 23, 27, 18, 13, 31, 12, 11, 23, 2, 25, 27, 20, 15, 3, 18, 21, 7, 28, 3, 31, 2, 1, etc. Finally, (12) follows from (37) on p. 225 of [3].

THE MAIN THEOREMS

Theorem 1

$L(n) \neq a^2 + b^2 + c^2$ iff $n \equiv 4, 8,$ or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$.

Proof: If $L(n) \neq a^2+b^2+c^2$, then (1) implies $L(n) = 4^j k$, with $j \geqslant 0$ and $k \equiv 7 \pmod 8$. (8) implies $j = 0$ or $j = 1$. Now (9) and (10) imply $n = 4, 8,$ or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$. Conversely, if $n \equiv 4$, 8, or $11 \pmod{12}$ or $n \equiv 21 \pmod{24}$, then (9) and (10) imply $L(n) \equiv 7 \pmod 8$ or $L(n) \equiv 28 \pmod{32}$, i.e., $L(n) = 4^j k$, with $j = 0$ or $j = 1$, and $k \equiv 7 \pmod 8$. Therefore, (1) implies $L(n) \neq a^2 + b^2 + c^2$.

Lemma 1

$$F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8 \text{ for } j \geqslant 2.$$

Proof: (Induction on j) If $j = 2$, then

$$F(12)/16 = 144/16 = 9 \equiv 1 \pmod 8.$$

Now let $j \geqslant 3$.

$$\frac{F(3 \star 4^j)}{4^{j+1}} = \frac{F(4 \star 3 \star 4^{j-1})}{4^{j+1}} = \frac{F(3 \star 4^{j-1})}{4^j} \quad \frac{L(3 \star 4^{j-1})L(6 \star 4^{j-1})}{4}$$

by (2). (11) implies $L(3 \star 4^{j-1}) \equiv 2 \pmod{32}$; (3) implies $L(6 \star 4^{j-1}) \equiv 2 \pmod{32}$. Thus

$$L(3 \star 4^{j-1})L(6 \star 4^{j-1}) \equiv 4 \pmod{32},$$

which implies $L(3 \star 4^{j-1})L(6 \star 4^{j-1})/4 \equiv 1 \pmod 8$. By the induction hypothesis, $F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8$. Therefore,

$$F(3 \star 4^j)/4^{j+1} \equiv 1 \star 1 \equiv 1 \pmod 8.$$

Lemma 2

$$F(3 \star 4^{j-1}m)/4^j \equiv m \pmod 8 \text{ for } j \geqslant 2 \text{ and } m \geqslant 0.$$

Proof: (Induction on m) Since $F(0) = 0$, Lemma 2 holds for $m = 0$. (4) implies

$$\begin{aligned} F(3 \star 4^{j-1}(m+1))/4^j &= F(3 \star 4^{j-1}m + 3 \star 4^{j-1})/4^j \\ &= \left(F(3 \star 4^{j-1}m)/4^j\right)F(3 \star 4^{j-1} - 1) \\ &\qquad + F(3 \star 4^{j-1}m + 1)\left(F(3 \star 4^{j-1})/4^j\right); \end{aligned}$$

by the induction hypothesis, $F(3 \star 4^{j-1}m)/4^j \equiv m \pmod 8$; (5) implies

$$F(3 \star 4^{j-1} - 1) \equiv F(3 \star 4^{j-1}m + 1) \equiv 1 \pmod 8;$$

Lemma 1 implies

$$F(3 \star 4^{j-1})/4^j \equiv 1 \pmod 8.$$

Therefore,

$$F(3 \star 4^{j-1}(m + 1))/4^j \equiv m \star 1 + 1 \star 1 \equiv m + 1 \pmod 8.$$

Theorem 2

$F(n) \neq a^2 + b^2 + c^2$ iff $n \equiv 10 \pmod{12}$ or $n = 3 \star 4^{j-1}m$, with $j \geqslant 2$ and $m \equiv 7 \pmod 8$.

Proof: If $F(n) \neq a^2 + b^2 + c^2$, then (1) implies $F(n) = 4^j t$ with $j \geqslant 0$ and $t \equiv 7 \pmod 8$. (7) implies $j \neq 1$. If $j = 0$, then (6) implies $n \equiv 10 \pmod{12}$. If $j \geqslant 2$, then (12) implies $n = 3 \star 4^{j-1}m$. Now Lemma 2 implies $m \equiv t \equiv 7 \pmod 8$. Conversely, if $n \equiv 10 \pmod{12}$, then (6) implies $F(n) \equiv 7 \pmod 8$, hence (1) implies $F(n) \neq a^2 + b^2 + c^2$. If $n = 3 \star 4^{j-1}m$ with $j \geqslant 2$ and $m \equiv 7 \pmod 8$, then (12) implies $F(n) = 4^j t$. Lemma 2 implies $t = F(n)/4^j \equiv m \pmod 8$. Since $t \equiv 7 \pmod 8$, (1) implies

$$F(n) \neq a^2 + b^2 + c^2.$$

REFERENCES

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