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FIBONACCI GRACEFUL GRAPHS

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(Submitted November 1980)

1. INTRODUCTION

A simple graph $G(p, n)$ with p vertices and n edges is *graceful* if there is a labeling ℓ of its vertices with distinct integers from the set

$$\{0, 1, 2, \ldots, n\}$$

so that the induced edge labeling ℓ', defined by

$$\ell'(uv) = |\ell(u) - \ell(v)|$$

assigns each edge a different label. The problem of characterizing all graceful graphs remains open (Golomb [3]), and in particular the Ringel-Kotzig-Rosa conjecture that all trees are graceful is still unproved after fifteen years. (For a summary of the status of this conjecture, see Bloom [2].) Other classes of graphs that are known to be graceful include complete bipartite graphs (Rosa [7]), wheels (Höede & Kuiper [5]), and cycles on n vertices where $n \equiv 0$ or 3 (mod 4) (Hebbare [4]).

A natural extension of the idea of a graceful graph is to have the induced edge labeling ℓ' of $G(p, n)$ be a bijection onto the first n terms of an arbitrary sequence of positive integers $\{a_i\}$. In a recent paper, Koh, Lee, & Tan [6] chose the sequence $\{a_i\}$ to be the Fibonacci numbers $\{F_i\}$ where

$$F_n = F_{n-1} + F_{n-2}; \; F_1 = F_2 = 1.$$

They defined a *Fibonacci tree* to be a tree $T(n + 1, n)$ in which the vertices can be labeled with the first $n+1$ Fibonacci numbers so that the induced edge numbers will be the first n Fibonacci numbers. Koh, Lee, & Tan gave a systematic way to obtain all Fibonacci trees as subgraphs of a certain class of graphs and showed that the number of (labeled) Fibonacci trees on $n+1$ vertices is equal to F_n. The only graphs other than trees which can be labeled in this fashion are certain unicyclic graphs where the cycle is of length three.

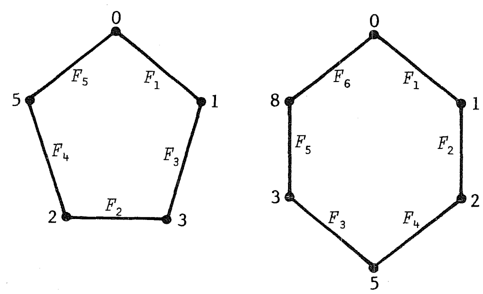
In this paper, we modify the definition of Koh, Lee, & Tan so that the vertex labels of $G(p, n)$ are allowed to be distinct integers (not necessarily Fibonacci numbers) from the set $\{0, 1, 2, 3, 4, \ldots, F_n\}$. Formally, we make the following:

**This work was supported by a grant from the Faculty Research Committee, University of Wisconsin-La Crosse.*

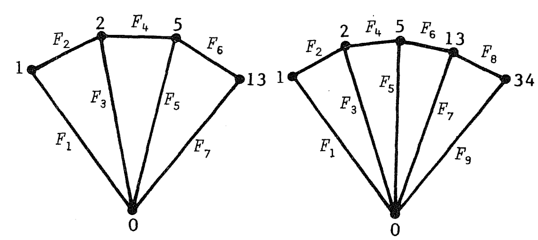
Definition

A graph $G(p, n)$ will be called *Fibonacci graceful* if there is a labeling ℓ of its vertices with distinct integers from the set $\{0, 1, 2, 3, 4, \ldots, F_n\}$ so that the induced edge labeling ℓ', defined by $\ell'(uv) = |\ell(u) - \ell(v)|$, is a bijection onto the set $\{F_1, F_2, F_3, \ldots, F_n\}$.

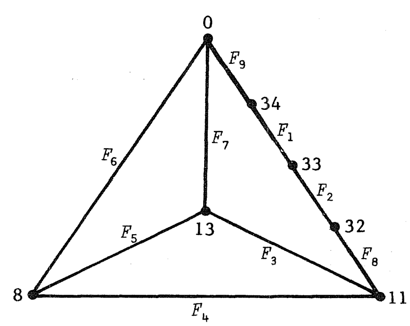
This definition gives rise to an extensive class of graphs that are Fibonacci graceful; several examples appear in Figure 1. In Sections 2 and 3, we shall show how the cycle structure of Fibonacci graceful graphs is determined by the properties of the Fibonacci numbers. In Sections 4 and 5, we shall prove that several classes of graphs are Fibonacci graceful, including almost all trees. The general question of characterizing all Fibonacci graceful graphs will remain open.



a. Cycles C_5 and C_6



b. Fans



c. A graph homeomorphic to K_4

FIGURE 1. SOME FIBONACCI GRACEFUL GRAPHS

2. SOME PROPERTIES OF FIBONACCI GRACEFUL GRAPHS

From the definition of a Fibonacci graceful graph, it is apparent that the edge numbered F_n must have 0 and F_n as the vertex numbers of its endpoints.

Furthermore, any vertex adjacent to the vertex labeled 0 must be labeled with a Fibonacci number. The remaining vertices receive integer labels between 0 and F_n, but these need not be Fibonacci numbers.

It is easy to see that if a graph is Fibonacci graceful, then it may have several distinct labelings. In fact, we have the standard "inverse node numbering" ([3], p. 27).

Observation 1: If $\{0 = a_1, a_2, a_3, \ldots, a_n = F_n\}$ is a set of vertex labels of a Fibonacci graceful graph, then changing each label a_i to $F_n - a_i$ also gives a Fibonacci graceful labeling of the graph.

We also have the following theorem which demonstrates that the cycle structure of Fibonacci graceful graphs is dependent on Fibonacci identities.

Theorem 1

Let $G(p, n)$ be a graph with a Fibonacci graceful labeling and let C_i be a cycle of length k in G. Then there exists a sequence $\{\delta_{ij}\}_{j=1}^{k}$ with $\delta_{ij} = \pm 1$ for all $j = 1, 2, \ldots, k$ such that

$$\sum_{j=1}^{k} \delta_{ij} F_{ij} = 0$$

where $\{F_{ij}\}_{j=1}^{k}$ are the Fibonacci numbers for the edges in C_i.

Proof: Let $a_1, a_2, \ldots, a_k$ be the vertex labels for C_i. Clearly,

$$\sum_{j=1}^{k-1} (a_{j+1} - a_j) + (a_1 - a_k) = 0.$$

Since each difference $a_{j+1} - a_j$ equals either an edge label on C_i or its negative, the theorem follows. ■

Corollary 1.1

If graph G has a Fibonacci graceful labeling, then the edges of any cycle of length 3 in G must be numbered with 3 consecutive Fibonacci numbers (note that F_1, F_3, F_4 is equivalent to F_2, F_3, F_4).

Corollary 1.2

If graph G has a Fibonacci graceful labeling, then the edges of any cycle of length 4 in G must be numbered with a sequence of the form $F_i, F_{i+1}, F_{i+3}, F_{i+4}$.

Corollary 1.3

If graph G has a Fibonacci graceful labeling, then the edges of any cycle of length 5 must be numbered with either a sequence of the form $F_i, F_{i+1}, F_{i+3}, F_{i+5}, F_{i+6}$ or $F_1, F_2, F_i, F_{i+1}, F_{i+2}$.

Corollary 1.4

Let graph G have a Fibonacci graceful numbering. Suppose that in cycle C_i of length k the three largest edge labels are consecutive Fibonacci numbers, F_{ik-2}, F_{ik-1}, F_{ik}. Then for the remaining labels on C_i we have

$$\sum_{j=1}^{k-3} \delta_{ij} F_{ij} = 0.$$

Proof: Both δ_{ik-2} and δ_{ik-1} must be opposite in sign to δ_{ik} for, otherwise, the sum of F_{ik} and either of F_{ik-2} or F_{ik-1} would exceed the sum of all the remaining edge labels on C_i, violating Theorem 1. [See Identity (2) below]. ■

For convenience, we list some of the basic Fibonacci identities that are useful later:

(1) $F_n = F_{n-1} + F_{n-2}$; $F_1 = F_2 = 1$.

(2) $F_1 + F_2 + F_3 + \cdots + F_{n-2} = F_n - 1$.

(3) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

(4) $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n} - 1$.

A variation of Identities (3) and (4) may be obtained by once omitting a pair of consecutive Fibonacci numbers:

(5) $F_n - 1 > F_{n-2} + F_{n-4} + F_{n-6} + \cdots + F_{j+2} + F_j + F_{j-3} + F_{j-5} + \cdots$, $(j \geqslant 3)$.

The next result, stated as a lemma, is useful both in seeking Fibonacci graceful labelings and in developing a theory of the structure of Fibonacci graceful graphs.

Lemma 1

Suppose $G(p, n)$ has a Fibonacci graceful labeling and C is a cycle of G.

a. If F_k is the largest Fibonacci number appearing as an edge label of C, then F_{k-1} also appears on C. In particular, the edge labeled F_{n-1} must be in every cycle that contains the edge labeled F_n.

b. The cycle C whose largest edge number is F_k must contain either the edge labeled F_{k-2} or F_{k-3}.

Proof:

a. By Theorem 1, some linear combination of the edge numbers on C must sum to 0. By Identity (2):

$$F_1 + F_2 + F_3 + \cdots + F_{k-2} = F_k - 1 < F_k.$$

Thus, F_{k-1} must appear as an edge label of C.

b. Since $F_k - F_{k-1} = F_{k-2}$, some combination of the remaining labels on C must equal F_{k-2}. But, since $F_1 + F_2 + \cdots + F_{k-4} < F_{k-2}$, there must be an edge labeled F_{k-3} unless there is one labeled with F_{k-2} itself. ∎

We also have the following theorem, which corresponds to a well-known result for graceful graphs [3, p. 26].

Theorem 2

If $G(p, n)$ is Eulerian and Fibonacci graceful, then $n \equiv 0$ or $2 \pmod 3$.

Proof: If G is Eulerian, then it can be decomposed into edge-disjoint cycles. By Theorem 1, the sum of the edge numbers around any cycle must be even and, hence,

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

must also be even. Thus, F_{n+2} must be odd, and this occurs if and only if $n \equiv 0$ or $2 \pmod 3$. ∎

3. FORBIDDEN SUBGRAPHS

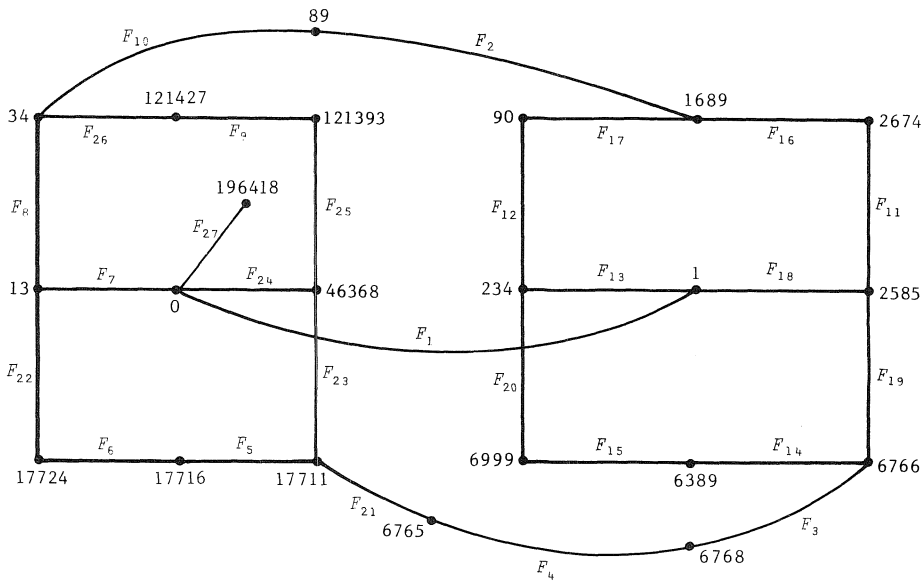
One possible way to characterize Fibonacci graceful graphs would be to find a complete list of graphs such that G would be Fibonacci graceful if and only if it did not contain a subgraph isomorphic to one on this list. This approach seems difficult because gracefulness is a global rather than a local condition. Nevertheless, the following theorems do limit the structure of Fibonacci graceful graphs considerably.

Theorem 3

If $G(p, n)$ contains a 3-edge-connected subgraph, then G is not Fibonacci graceful.

Proof: Suppose $G(p, n)$ is Fibonacci graceful, and G' is a 3-edge connected subgraph. Let F_k be the largest edge number appearing in G', and let v_1 and v_2 be the endpoints of that edge. Since G' is 3-edge connected, there is a path joining v_1 and v_2 which does not contain either the edge numbered F_k or the edge numbered F_{k-1}. This path, together with the edge (v_1, v_2) forms a cycle which contains the edge labeled F_k, but not the one labeled F_{k-1}. This violates Lemma 1. ∎

It is interesting to note that a graph G which is not Fibonacci graceful may have homeomorphic copies thich are. For example, although K_4 is not Fibonacci graceful by Theorem 3, the graph in Figure 1(c), a homeomorphic copy of K_4, is Fibonacci graceful. Perhaps a more striking example is the nonplanar graph shown in Figure 2, which is Fibonacci graceful even though the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not. The graph in Figure 2 contains a subgraph homeomorphic to $K_{3,3}$. A consequence of the next theorem is that it is impossible for a nonplanar graph to contain a subgraph which is homeomorphic to K_5 and have a Fibonacci graceful labeling.



($K_{3,3}$ is homeomorphic with a subgraph containing the vertices labeled 24, 17711, 0 and 13, 46368, 1.)

FIGURE 2. A NONPLANAR FIBONACCI GRACEFUL GRAPH

Theorem 4

If there is a pair of vertices joined by 4 edge-disjoint paths in $G(p, n)$, then G is not Fibonacci graceful.

Proof: Let v_1 and v_2 be two vertices of G joined by 4 edge-disjoint paths P_1, P_2, P_3, and P_4. Suppose G has a Fibonacci graceful labeling. With no loss of generality, assume that F_k is the largest Fibonacci number on these paths and that it lies on an edge of P_1. By Lemma 1(a), F_{k-1} must also lie on P_1, since otherwise there are cycles containing edge F_k, but not F_{k-1}. Additionally, either F_{k-2} or F_{k-3} must also be an edge label on P_1, for if they were on other paths, say P_2 and/or P_3, then paths P_1 and P_4 would form a cycle violating Lemma 1(b).

Suppose that it is F_{k-2} that appears as an edge label on P_1. Then Corollary 1.4 permits us to ignore F_k, F_{k-1}, and F_{k-2} and tells us that some linear combination of the remaining Fibonacci numbers on any cycle must sum to 0. Repeat this process, beginning with the largest of the remaining edge labels, to discard or ignore the presence of three consecutive edge numbers on any of the paths. This repetition cannot discard all of the edge numbers along any path, for then vertices v_1 and v_2 would necessarily have the same vertex label. Thus, the process terminates at a time where F_j is the largest remaining edge label and F_j, F_{j-1}, and F_{j-3} appear on some path, say P_2, but F_{j-2} appears on another path, say P_4. Then there is a cycle, P_3 and P_4 for example, on which F_{j-2} is the largest Fibonacci number, but F_{j-3} does not appear, violating Lemma 1.

4. CLASSES OF FIBONACCI GRACEFUL GRAPHS

We begin with easy observations that any Fibonacci graceful graph may be embedded in larger ones.

Observation 2: Let $G(p, n)$ have a Fibonacci graceful labeling. Then the graph $G_1(p+1, n+1)$ formed from G by attaching a vertex v of degree 1 at the vertex labeled 0 can be given a Fibonacci graceful labeling by labeling v with F_{n+1}.

Observation 3: Let $G(p, n)$ have a Fibonacci graceful labeling. Then the graph $G_2(p+1, n+2)$ formed from G by attaching a vertex v of degree 2 to the vertices labeled 0 and F_n can be given a Fibonacci graceful labeling by numbering v with F_{n+2}.

Theorem 5

The cycle C_n is Fibonacci graceful if and only if $n \equiv 0$ or 2 (mod 3).

Proof: Since C_n is Eulerian, it is not Fibonacci graceful for $n \equiv 1$ (mod 3) by Theorem 3.

If $n \equiv 0$ (mod 3), the following labeling sequence on the vertices is a Fibonacci graceful labeling:

$$0,\ F_n,\ F_{n-2},\ F_{n-1},\ \ldots,\ F_{n-3j},\ F_{n-3j-2},\ F_{n-3j-1},\ \ldots,\ F_6,\ F_4,\ F_5,\ F_3,\ F_1.$$

If $n \equiv 2$ (mod 3), the following numbering sequence on the vertices is a Fibonacci graceful labeling:

$$0,\ F_n,\ F_{n-2},\ F_{n-1},\ \ldots,\ F_{n-3j},\ F_{n-3j-2},\ F_{n-3j-1},\ \ldots,\ F_5,\ F_3,\ F_4,\ F_1.\ \blacksquare$$

Theorem 6

A maximal outerplanar graph G with at least four vertices is a Fibonacci graceful graph if and only if it has exactly two vertices of degree 2.

Proof: Let G be a maximal outerplanar graph with more than two vertices of degree 2. Then G must contain a subgraph isomorphic to the graph shown in Figure 3. Since there are 4 edge-disjoint paths between vertices v_1 and v_2 in this graph, G cannot be Fibonacci graceful by Theorem 4.

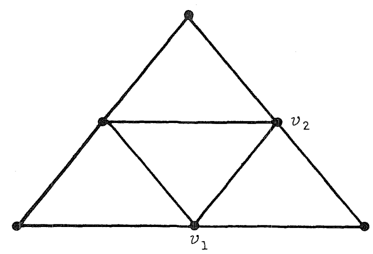


FIGURE 3. A FORBIDDEN SUBGRAPH

We next use induction to show that a maximal outerplanar graph $G(p,\ 2p-3)$ with exactly two vertices of degree 2 has a Fibonacci graceful labeling. Moreover, this labeling can be given so that the 0 label appears on either vertex of degree 2, say v_0, and that F_{2p-3} may be the label of either neighbor of v_0. Since all the maximal outerplanar graphs with two vertices of degree 2 can be generated by repeatedly adjoining a new vertex of degree 2 to a previous vertex of degree 2 and one of its neighbors ([1], p. 607), Observation 3 will complete the proof.

To begin the induction and to illustrate the labeling, Figure 4 shows all the maximal outerplanar graphs with exactly two vertices of degree 2 for $p =$ 4, 5, and 6. Assume the inductive hypothesis is valid for $p = k$ and consider a maximal outerplanar graph $G(p+1,\ 2p-1)$ with exactly two vertices of degree 2. Let v_0 be a vertex of degree 2 in G with neighbors v_1 and v_2. When v_0 is removed, one of its neighbors, say v_1, will become a vertex of degree 2 in $G - v_0$. By induction, $G - v_0$ may be given a vertex labeling ℓ such that

$$\ell(v_1) = 0 \quad \text{and} \quad \ell(v_2) = F_{2p-3}.$$

By Observation 3, G can be made Fibonacci graceful by labeling v_0 with F_{2p-1}. By Observation 1, the transformation $F_{2p-1} - a_i$ applied to the vertex labels gives G a Fibonacci graceful labeling ℓ_1 with

$$\ell_1(v_0) = 0 \quad \text{and} \quad \ell_1(v_1) = F_{2p-1}.$$

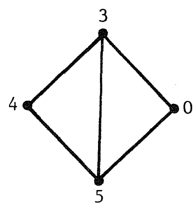
To show that G has a second labeling ℓ_2 in which

$$\ell_2(v_2) = F_{2p-1},$$

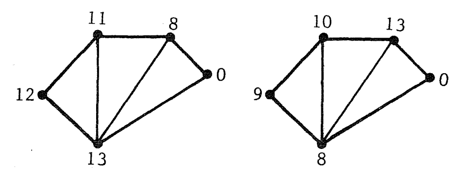
apply the transformation $F_{2p} - a_i$ to the vertex labels of $G - v_0$. This gives an induced edge labeling ℓ_2' to G for which

$$\ell_2'(v_0v_1) = F_{2p-2} \quad \text{and} \quad \ell_2'(v_0v_1) = F_{2p-1}$$

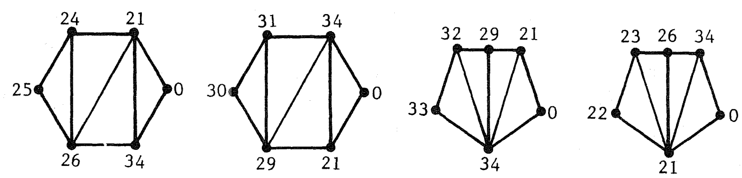
with all other edge labels unchanged. ■



a. $p = 4$



b. $p = 5$



c. $p = 6$

FIGURE 4. FIBONACCI GRACEFUL LABELINGS OF MAXIMAL OUTERPLANAR GRAPHS WITH SIX OR FEWER VERTICES AND EXACTLY TWO VERTICES OF DEGREE 2

5. FIBONACCI GRACEFUL TREES

In this section we will present an algorithm that will enable one to find a Fibonacci graceful labeling for nearly all trees. The trees which do not have such a labeling are easily characterized. Except for K_1 and K_2, which are trivially labeled, any tree $T(p, n)$ with five or fewer vertices cannot be Fibonacci graceful since with $n \leqslant 4$ edges there are not enough distinct integers between 0 and F_n to label the $p = n+1$ vertices of T. It is also apparent that $K_{1,n}$ is not Fibonacci graceful for $n \geqslant 2$. That this is so follows from the fact that all the edges have a vertex in common and if the remaining vertices are distinctly labeled, there cannot be two edges with the label 1. The only other tree that is not Fibonacci graceful is shown in Figure 5.

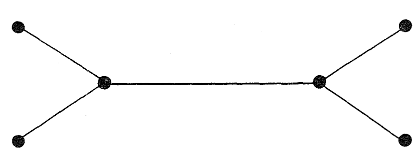


FIGURE 5. A TREE THAT IS NOT FIBONACCI GRACEFUL

It is generally easy to provide a labeling for any other tree, especially one with a large number of vertices, because for n large, F_n is considerably larger then $n + 1$ and there are many distinct integers from which to choose the vertex labels. In Figure 6 we show a Fibonacci graceful labeling for the remaining trees with six vertices.

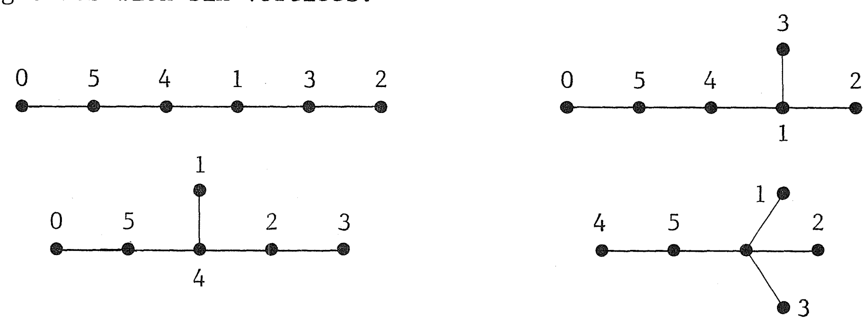


FIGURE 6. THE FIBONACCI GRACEFUL TREES $T(6, 5)$

The trees in Figure 6 are examples of a class of trees called "caterpillars"—trees which become paths when all of their endpoints are removed. (It is known that all caterpillars are graceful trees [8].) The *length* of a caterpillar will be the number of edges in the remaining path.

Theorem 7

All trees $T(n+1, n)$ with $n \geqslant 6$, except for $K_{1,n}$, are Fibonacci graceful.

Proof: We divide the proof into cases, and provide a labeling ℓ for each case. The cases are:

a. caterpillars of length 1;

b. caterpillars of length 2 or more;

c. noncaterpillars.

We begin with caterpillars of length 1. Since T has at least six edges, there is a vertex v_0 of T with degree 4 or more. Let v_1 denote the neighbor of v_0 which is not an endpoint. Let ℓ label v_0 with 0; v_1 with F_n; the $k + 1 \geqslant 3$ endpoints adjacent to v_0 with $1, F_{n-1}, F_{n-2}, \ldots, F_{n-k}$; and the endpoints adjacent to v_1 with $F_n - F_{n-k-1}, F_n - F_{n-k-2}, \ldots, F_n - 3, F_n - 2, F_n - 1$. Figure 7 gives an example of the results of this procedure. Clearly the algorithm gives a proper edge labeling; thus, it remains only to verify that the vertex labels are distinct. Note that, if v_i is a neighbor of v_0 and v_j is a neighbor of v_1, then $\ell(v_j) > \ell(v_i)$ since for $n \geqslant 6$ and $2 \leqslant k \leqslant n - 3$ we have:

$$\min\{\ell(v_j)\} = F_n - F_{n-k-1} > F_{n-k} = \max\{\ell(v_i)\}.$$

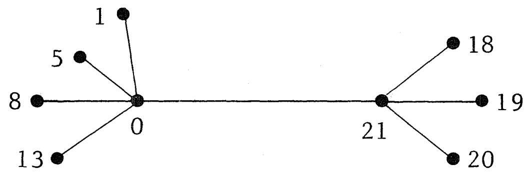


FIGURE 7. A FIBONACCI GRACEFUL CATERPILLAR OF LENGTH 1

For a caterpillar T of length 2 or more, choose a longest path in T and call its vertices $v_0, v_1, v_2, \ldots, v_k$. Denote the endpoints adjacent to v_i by $v_{i1}, v_{i2}, \ldots, v_{ij}$, $i = 1, 2, \ldots, k$. We consider two subcases depending on the degree of v_1. If v_1 is of degree 2, define ℓ as follows. Let

$$\ell(v_0) = 0,\ \ell(v_1) = F_n,\ \ell(v_2) = F_n - 1.$$

Then label the neighbors of v_2 by

$$\ell(v_{21}) = \ell(v_2) - F_{n-1},\ \ell(v_{22}) = \ell(v_2) - F_{n-2},\ \ldots,\ \ell(v_{2j}) = \ell(v_2) - F_{n-j},$$

and, finally,

$$\ell(v_3) = \ell(v_2) - F_{n-j-1}.$$

Proceed to define for the $r + 1$ neighbors of v_3,

$$\ell(v_{31}) = \ell(v_3) + F_{n-j-2},\ \ell(v_{32}) = \ell(v_3) + F_{n-j-3},\ \ldots,$$

$$\ell(v_{3r}) = \ell(v_3) + F_{n-j-r-1},$$

ending with

$$\ell(v_4) = \ell(v_3) + F_{n-j-r-2}.$$

Notice that each neighbor of v_3 has been distinctly labeled with positive integers strictly between $\ell(v_2)$ and $\max\{\ell(v_3), \ell(v_{2i})\}$. For the neighbors of v_4 label each vertex with

$$\ell(v_4) - \text{(the appropriate Fibonacci number)}.$$

Again each of these will be distinctly labeled with positive integers between $\ell(v_3)$ and $\min\{\ell(v_4), \ell(v_{3i})\}$. Continue in this manner, *adding* the continuing sequence of Fibonacci numbers to the neighbors of $v_5, v_7, v_9, \ldots$ and *subtracting* them from the neighbors of $v_6, v_8, v_{10}, \ldots$. An example of the resulting labels is shown in Figure 8(a).

If vertex v_1 is of degree more than 2, let

$$\ell(v_0) = 0 \quad \text{and} \quad \ell(v_1) = F_n$$

as before. For the neighbors of v_1, define

$$\ell(v_{11}) = F_n - 1,\ \ell(v_{12}) = \ell(v_1) - F_{n-1},\ \ell(v_{13}) = \ell(v_1) - F_{n-2},\ \ldots,$$

$$\ell(v_{1j}) = \ell(v_1) - F_{n-j-2},$$

ending with

$$\ell(v_2) = \ell(v_1) - F_{n-j-2}.$$

Proceed to label the neighbors of v_2 by *adding* the appropriate sequence of Fibonacci numbers to $\ell(v_2)$. In this instance, the vertex labels for these vertices will lie between $\ell(v_{11})$ and $\ell(v_2)$, the two largest vertex labels appearing on the neighbors of v_1. From here, proceed in a fashion analogous to that above. An example of such a caterpillar is shown in Figure 8(b).

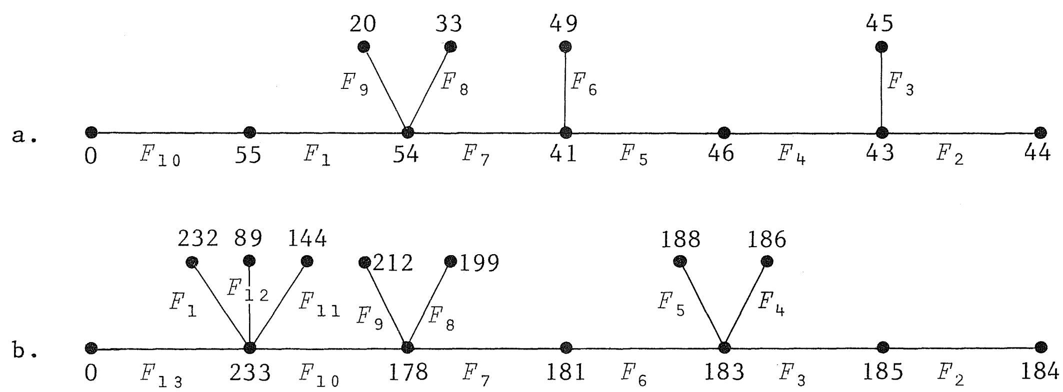


FIGURE 8. TWO LABELED CATERPILLARS OF LENGTH 4

Finally, we consider a tree T which is not a caterpillar. Remove the two endpoints of a longest path in T to form a subtree T' that is not a path. T' has either one or two centers, both lying on some longest path P' in T'. Select one of the centers, denoted v_0, and root T' at v_0. If v_0 is a vertex of degree $k \geqslant 2$, denote the neighbors of v_0 by $v_{11}, v_{12}, \ldots, v_{1k}$ in such a way that v_{11} and v_{1k} lie on P' and v_{1k} is the other center of T' if there are two centers. Denote the "half" of P' containing v_0 and v_{11} by P_L' (the "left half") and the section containing v_0 and v_{1k} by P_R' (the "right half"). (Thus, the vertices at the first level are labeled from left to right.) Also denote the k subtrees rooted at $v_{11}, v_{12}, \ldots, v_{1k}$ by $T_1, T_2, \ldots, T_k$, respectively. Next call the vertices at a distance of 2 from v_0 by $v_{21}, v_{22}, \ldots, v_{2j}$ in such a way that v_{21} is on P_R' and v_{2j} is on P_L'; that is, name the vertices from right to left. Proceed to name the vertices at distance 3, $v_{31}, v_{32}, \ldots, v_{3r}$ again from right to left. Continue from right to left at each level until all the vertices of T' have been named. Note that there will be at least two vertices at each distance or level (except perhaps at the final level, where there may be only a single vertex on P_R'), since v_0 was a center. Also, there must be a level with at least three vertices, since T' is not a path.

We define the Fibonacci graceful labeling ℓ on T' as follows:

$$\ell(v_0) = 0;$$

$$\ell(v_{11}) = F_n,\ \ell(v_{12}) = F_{n-1},\ \ldots,\ \ell(v_{1k}) = F_{n-k-1};$$

$$\ell(v_{21}) = \ell(v_{1k}) - F_{n-k-2};$$

$$\ell(v_{22}) = \ell(\text{parent vertex of } v_{22}) - F_{n-k-3},\ \ldots;$$

that is, for any subsequent vertex in T', its label will be the difference between the label of its parent vertex and the next smaller Fibonacci number. Note that the edges of T' receive the labels $F_n, F_{n-1}, \ldots, F_3$ in decreasing order from left to right on the first level, and from right to left on all subsequent levels. To extend ℓ to the original tree T, label each of the two endpoints which were removed by ℓ (its neighbor) - 1. Figure 9 presents two applications of this algorithm.

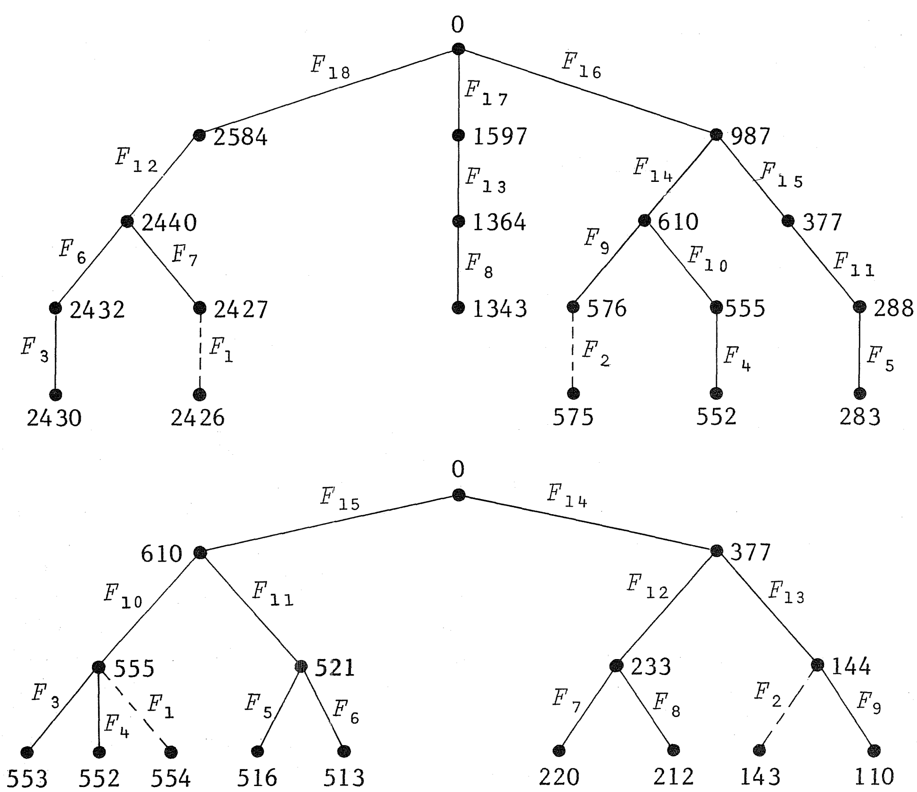


FIGURE 9. TREES WITH FIBONACCI GRACEFUL LABELINGS

It is clear that this procedure will properly label all the edges, so it remains only to observe that the vertex labels are distinct and nonnegative.

First, we note that within any of the rooted subtrees T_i, $i = 1, \ldots, k$, the vertex labels decrease as the distance from v_0 increases. Finally, we claim that for $i < j$, every vertex label in T_i exceeds those in T_j. Note that the vertex labels in T_1 all equal

$$F_n - (\text{a sum of Fibonacci numbers}),$$

where the terms in this sum include *at most*

$$F_{n-3}, F_{n-5}, F_{n-7}, \ldots, F_{n-r}, F_{n-r-3}, F_{n-r-5}, \ldots,$$

for some r, since at each level there is at least one edge in P_R, and at some level there is at least some other edge not on P. Thus, by Identity (5), the smallest vertex number in T_1 is greater than

$$F_n - (F_{n-2} - 1) > F_{n-1}.$$

Thus, every vertex number in T_1 exceeds any vertex number in T_2. A similar argument will show that if $v \in T_2$ ($\neq T_k$), then

$$F_{n-2} < \ell(v_2) \leqslant F_{n-1},$$

and that if $v \in T_k$, then

$$0 < \ell(v) \leqslant F_{n-k}.$$

This concludes the proof of the theorem. ■

6. SUMMARY AND CONCLUSION

In this paper, we have extended the idea of graceful graphs to numberings where the vertex labels are distinct integers but the edge labels are members of the Fibonacci sequence. We investigated the cycle structure of Fibonacci graceful graphs and used this to find forbidden subgraphs. We found infinite classes of Fibonacci gracegul graphs, including almost all trees. It is interesting to note that, if we had required the edge numbers of $T(n + 1, n)$ to come from the set $\{F_2, F_3, \ldots, F_{n+1}\}$ in order to eliminate the problem with duplicate vertex labels in $K_{1,n}$, then all trees could be labeled eadily. This is due to the large size of F_n relative to n, which leaves many possible distinct integers available for the vertex labels. Thus, in a certain sense, the Ringel-Kotzig-Rosa conjecture is a limiting case for this type of tree labeling problem, since to produce the edge labels $\{1, 2, 3, \ldots, n\}$ it is required to use every integer in $\{0, 1, 2, \ldots, n\}$.

For the Fibonacci graceful graphs, the problem remains to characterize all of them, perhaps by forbidden subgraphs, although this appears difficult in view of Observations 2 and 3. Further classes of Fibonacci graceful graphs can certainly be discovered. For example, we conjecture that all unicyclic graphs with at least one endpoint are Fibonacci graceful graphs.

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[continued from page 173]

Since the Newton iterates always fulfill the *linear* equations which belong to the system of nonlinear equations that is to be solved (with the exception, of course, of the starting value), the conclusion follows at once.

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