◆◇◆◇◆

PASCAL GRAPHS AND THEIR PROPERTIES*

NARSINGH DEO and MICHAEL J. QUINN
Washington State University, Pullman, WA 99164

(Submitted March 1982)

1. INTRODUCTION

While searching for a class of graphs with certain desired properties to be used as computer networks, we have found graphs that come close to being optimal. One of the desired properties is that the design be simple and recursive, so that when a new node is added, the entire network does not have to be reconfigured. Another property is that one central vertex be adjacent to all others. The third requirement is that there exist several paths between each pair of vertices (for reliability) and that some of these paths be of short lengths (to reduce communication delays). Finally, the graphs should have good cohesion and connectivity [1]. Complete graphs K_n satisfy all these properties, but are ruled out because of the expense.

This paper introduces a set of adjacency matrices called Pascal matrices, which are constructed using Pascal's triangle modulo 2. We also define Pascal graphs, the set of graphs corresponding to the Pascal matrices. We begin by showing that the Pascal graphs have the properties described above. In the second part of the paper we explore the properties of the determinants of the Pascal matrices. It appears that every Pascal matrix of order $\geqslant 3$ has a determinant of either 0 or 2. We indicate the sequence of matrix orders for which the determinant is 2. The third part of our report lists unexplored ideas and presents attributes of Pascal graphs which we have not been able to exploit in our proofs.

Standard graph theoretic terms are used throughout this paper. The reader seeking a reference should consult Deo [3] or Harary [6].

2. DEFINITIONS

Definition 1

An $n \times n$ symmetric binary matrix is called the *Pascal matrix* $PM(n)$ of order n if its main diagonal entries are all 0's and its lower triangle (and therefore the upper also) consists of the first $n - 1$ rows of the Pascal triangle modulo 2. Let $pm_{i,j}$ denote the element in the ith row and the jth column of the Pascal matrix.

**This work was supported by NSF grant MCS 78-25851 and by U.S. D.O.T. contract DTRS-5681-c-00033.*

(This definition should not be confused with another definition of Pascal matrix by Lunnon [8]. Note, however, that the matrix he defines as the Pascal matrix has been defined previously as *Tartaglia's rectangle* [9].)

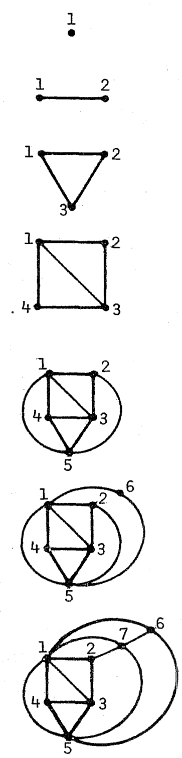
Definition 2

An undirected graph with n vertices corresponding to $PM(n)$ as its adjacency matrix is called the *Pascal graph* $PG(n)$ of order n.

The first seven Pascal graphs along with associated Pascal matrices are shown in Figure 1.

Definition 3

Let $pt_{i,j}$ refer to the jth element of the ith row of Pascal's triangle, where rows and their elements are numbered beginning with 0.



$PG(1)$ $PM(1)$ $[0]$

$PG(2)$ $PM(2)$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$PG(3)$ $PM(3)$ $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$PG(4)$ $PM(4)$ $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$PG(5)$ $PM(5)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

$PG(6)$ $PM(6)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$PG(7)$ $PM(7)$ $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

FIGURE 1

3. CONNECTIVITY PROPERTIES OF THE PASCAL GRAPHS

Lemma 1

$PG(n)$ is a subgraph of $PG(n + 1)$ for all $n \geqslant 1$.

Proof: This property is a direct consequence of the definition of the Pascal matrix

Theorem 1

All $PG(i)$ for $1 \leqslant i \leqslant 7$ are planar; all Pascal graphs of higher order are nonplanar.

Proof: Figure 1 clearly indicates that all $PG(i)$ for $1 \leqslant i \leqslant 7$ are planar. $K_{3,3}$ is a subgraph of $PG(8)$. Thus, by Lemma 1, all graphs of order 8 and higher are nonplanar.

Theorem 2

Vertex v_1 is adjacent to all other vertices in the Pascal graph. Vertex v_i is adjacent to v_{i+1} in the Pascal graph for $i \geqslant 1$.

Proof: $pm_{i,j} = pt_{i-2,j-1} \pmod 2$, $i > j \geqslant 1$ (Definition of Pascal matrix).

For all $i \geqslant 2$, $pm_{i,1} = pt_{i-2,0} \pmod 2 = \binom{i-2}{0} \pmod 2 = 1$.

Thus, v_1 is adjacent to all v_i, $i \geqslant 2$.

For all $i \geqslant 1$, $pm_{i+1,i} = pt_{i-1,i-1} \pmod 2 = \binom{i-1}{i-1} \pmod 2 = 1$.

Thus, v_i is adjacent to v_{i+1} for all $i \geqslant 1$.

Corollary 1

$PG(n)$ contains a startree for all $n \geqslant 1$.

Corollary 2

$PG(n)$ contains a Hamiltonian circuit $[1, 2, \ldots, n - 1, n, 1]$.

Corollary 3

$PG(n)$ contains $W_n - x$ (wheel of order n minus an edge).

Lemma 2

If $k = 2^n + 1$, n a positive integer, then v_k is adjacent to all v_i, $1 \leqslant i < 2k$ and $i \neq k$.

Proof: Let $k = 2^n + 1$, where n is a positive integer.

Case 1. $1 \leqslant i < k$

$$pm_{k,i} = pt_{k-2,i-1} \pmod 2 = \binom{2^n - 1}{i - 1} \pmod 2 = 1 \quad [4].$$

Case 2. $k < i < 2k$

$$pm_{k,i} = pm_{i,k} = pt_{i-2,k-1} \pmod 2 = \binom{i-2}{2^n} \pmod 2.$$

We may factor $i - 2$ into its binomial coefficients:

$$i - 2 = m_0 + m_1 \times 2^1 + \cdots + m_{n-1} \times 2^{n-1} + 1 \times 2^n.$$

Thus,

$$\binom{i-2}{2^n} \pmod 2 = \binom{m_0}{0}\binom{m_1}{0} \cdots \binom{m_{n-1}}{0}\binom{1}{1} \pmod 2 = 1 \quad [4].$$

Since for all v_i, $1 \leqslant i < 2k$ and $i \neq k$, $pm_{k,i} = 1$, v_k is adjacent to all such v_i.

The following connectivity property is useful in the design of reliable communication and computer networks.

Theorem 3

There are at least two edge-disjoint paths of length $\leqslant 2$ between any two distinct vertices in $PG(n)$, $n \geqslant 3$.

Proof: Let v_i, v_j be two vertices of $PG(n)$, $n \geqslant 3$, $i < j$.

Case 1. $i = 1$, $j = 2$

Two edge-disjoint paths are $[v_1, v_2]$ and $[v_1, v_3, v_2]$.

Case 2. $i = 1$, $j > 2$

Two edge-disjoint paths are $[v_1, v_i]$ and $[v_1, v_{i-1}, v_i]$ (Lemma 2).

Case 3. $i > 1$

By Theorem 2, we know that one path is $[v_i, v_1, v_j]$. Let

$$k = 1 + 2^{\lfloor \log_2(j) \rfloor}.$$

Lemma 2 indicates that v_k is adjacent to all v_m where $1 \leqslant m < 2k$ and $m \neq k$. If $i = k$ or $j = k$, then a second path is $[v_i, v_j]$; otherwise, a second path is $[v_i, v_k, v_j]$.

Corollary 4

All Pascal graphs of order $\geqslant 3$ are 2-connected.

Lemma 3

No two even-numbered vertices of a Pascal graph are adjacent.

Proof: Let i, j be even integers, $i > j$.

$$pm_{i,j} = pt_{i-2,j-1} \pmod 2 = \binom{i-2}{j-1} \pmod 2.$$

Since $i - 2$ is even and $j - 1$ is odd, $\binom{i-2}{j-1} \pmod 2 = 0$ [4].

Theorem 4

If v_i is adjacent to v_j, where j is even and $|i - j| > 1$, then i is odd and v_i is adjacent to v_{j-1}.

Proof: Assume v_i is adjacent to v_j, where j is even and $|i - j| > 1$. By Lemma 3, we know that i is odd.

Case 1. $i > j$

$1 = pm_{i,j} = pm_{i-1,i} + pm_{i-1,j-1} \pmod 2$ (Definition of Pascal triangle)

$= 0 + pm_{i-1,j-1}$ (Lemma 3)

$= pm_{i-1,j-1}$.

Thus,

$pm_{i,j-1} = pm_{i-1,j-1} + pm_{i-1,j-2} \pmod 2$ (Definition of Pascal triangle)

$= pm_{i-1,j-1} + 0$ (Lemma 3)

$= 1$ (Above).

Case 2. $i < j$

The proof proceeds similarly to Case 1. Thus, since $pm_{i,j-1} = 1$, v_i is adjacent to v_{j-1}.

Although the set of complete graphs K_n has maximal connectivity and cohesion properties, the fact that the number of edges in K_n increases at a rate of n^2 makes it too costly to consider. The following theorem shows that the number of edges in the Pascal graphs increases at a much lower rate.

Theorem 5

Define $e(PG(n))$ to be the number of edges in $PG(n)$. Then

$$e(PG(n)) \leqslant \lfloor (n-1)^{\log_2 3} \rfloor.$$

Proof by Induction:

Basis

$$e(PG(1)) = 0 \leqslant 1 = \lfloor 0^{\log_2 3} \rfloor.$$

$$e(PG(2)) = 1 \leqslant 1 = \lfloor 1^{\log_2 3} \rfloor.$$

$$e(PG(3)) = 3 \leqslant 3 = \lfloor 2^{\log_2 3} \rfloor.$$

Induction

Assume true for all $PG(n)$, $1 \leqslant n \leqslant 2^k + 1$, $k > 0$.

Prove true for $PG(n)$, $2^k + 2 \leqslant n \leqslant 2^{k+1} + 1$.

Let r be the positive integer such that $n - 1 = 2^k + r$.

$$\begin{aligned} e(PG(n)) &= e(PG(2^k + 1)) + 2e(PG(r + 1)) && [7] \\ &\leqslant \lfloor (2^k)^{\log_2 3} \rfloor + 2\lfloor r^{\log_2 3} \rfloor && \text{(Induction Hypothesis)} \\ &\leqslant \lfloor (2^k + r)^{\log_2 3} \rfloor = \lfloor (n - 1)^{\log_2 3} \rfloor. \end{aligned}$$

Pascal graphs are not the graphs with the fewest possible edges satisfying the preceding structural properties (which are useful in designing practical networks). For example, in $PG(7)$, the edge from v_2 to v_7 is redundant. There is a possibility that for some set of connectivity requirements, the Pascal graphs may exhibit optimal connectivity; i.e., they have no redundant edges. We have not found such a set of requirements, however.

4. DETERMINANTS OF THE PASCAL MATRICES

Theorem 6

Let $\det(PM(n))$ refer to the determinant of the Pascal matrix of order n. Then $\det(PM(n)) = 0$ for all even $n \geqslant 4$.

Proof: Given $PM(n)$ satisfying the conditions on n, we show that the even-numbered rows of $PM(n)$ are linearly dependent.

No two even-numbered vertices of a Pascal graph are adjacent (Lemma 3). Since even-numbered vertices are only adjacent to odd-numbered vertices, and since we desire to show that the even-numbered rows are linearly dependent, we may create a *reduced Pascal matrix* by removing the odd-numbered rows and even-numbered columns from $PM(n)$ (see Figure 2).

To show that the even-numbered rows are linearly dependent, it is sufficient to show that the determinant of the reduced Pascal matrix is 0. The reduced Pascal matrix contains two columns of 1's. Vertex v_1 is adjacent to all the other vertices (Theorem 2). Let $k = 2^{\lfloor \log_2 (n-1) \rfloor} + 1$. Vertex v_k of $PG(n)$ is adjacent to all other v_i, $1 \leqslant i < k$ or $k + 1 < i < 2k$ (see Lemma 2). Thus, columns 1 and $(\lfloor k/2 \rfloor + 1)$ of the reduced Pascal matrix consist only of 1's.

Since the reduced Pascal matrix contains two identical columns, its determinant is 0. Thus $\det(PM(n)) = 0$.

	1	2	3	4	5	6	7	8
1	0	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0
3	1	1	0	1	1	0	0	1
4	1	0	1	0	1	0	0	0
5	1	1	1	1	0	1	1	1
6	1	0	0	0	1	0	1	0
7	1	1	0	0	1	1	0	1
8	1	0	1	0	1	0	1	0

Old row	New row	1	3	5	7	Original column
		1	2	3	4	New column
2	1	1	1	1	1	
4	2	1	1	1	0	
6	3	1	0	1	1	
8	4	1	1	1	1	

FIGURE 2. THE REDUCED PASCAL MATRIX

Theorem 7

$\mathrm{Det}(PM(n))$ is even for all odd $n \geqslant 3$.

Proof: Let n be an odd integer $\geqslant 3$; G_i be one of the m linear subgraphs of $PG(n)$; e_i be the number of components of G_i which have an even number of vertices; and c_i be the number of cycles in G_i.

$$\mathrm{Det}(PM(n)) = \sum_{i=1}^{m} (-1)^{e_i} \times 2^{c_i} \qquad [5].$$

Since there are an odd number of vertices, each linear subgraph of $PG(n)$ must contain at least one cycle. Thus, $\det(PM(n))$ is a sum of even integers, and therefore $\det(PM(n))$ is even.

Observations

$$\operatorname{Det}(PM(n)) = \begin{cases} 2, & \text{for } n = 3, 7, 11, 23, 43, 87 \\ 0, & \text{for all other } n,\ 4 \leqslant n \leqslant 86. \end{cases}$$

Let $t_0, t_1, t_2, \ldots$ be the sequence of integers such that $\det(PM(t_i)) = 2$. Then the sequence of t_i's is conjectured to be:

$$t_0 = t_1 = 3$$

$$t_i = 2^i + t_{i-2},\ i \geqslant 2.$$

$\operatorname{Det}(PM(t_i + 1)) = 0$ for all i, since $t_i + 1$ is even. This implies that row $t_i + 1$ is linearly dependent upon other even-numbered rows in the Pascal matrix (Theorem 6). It appears that the first of these rows whose linear combination yields row $t_i + 1$ is row $t_{i-1} - 1$. This linear combination of rows must always break down at column $2^{i+1} + t_{i-1}$, since this column has a 1 in row $t_{i-1} - 1$ and 0's in rows $t_{i-1} + 1$ through $t_i + 1$. Note that it is precisely at this point, when the linear dependence must break down, that the Pascal matrix again has determinant 2. Figure 3 illustrates this phenomenon.

$$\begin{array}{r|ccccccccccccccccccccccc}
 & & & & & & & & & & & & & & & & 2^{i+1} & & & & & & & 2^{i+1}+t_{i-1} \\
\hline
 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
t_{i-1} - 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_i & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_i + 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

FIGURE 3

Thus discussion leaves several questions unanswered. We just described why the linear combination of rows breaks down when it does. Why does it fail to break down sooner? When it does break down, why is there not another combination of linearly dependent rows? Why is the determinant of $PM(t_i)$, $i \geqslant 0$, equal to 2?

There is a pattern to the rows that are linearly dependent on each other, causing the determinants of the matrices to be 0. Relationships among these rows are illustrated in Figure 4.

This combination of rows...	yields row...	for Pascal matrices of size...
2	4	4-6
2	8	8-10
-6 +8 +10	12	12-22
-10 +16 +18	24	24-42
22 -24 -26 +28 -38 +40 +42	44	44-86

For example, row 8 plus row 10 minus row 6 equals row 12 in all Pascal matrices of sizes 12 through 22. Thus, since row 12 is linearly dependent upon other rows, the determinant of each Pascal matrix of order 12 through 22 is zero.

Note that $6 = 4 + 2$, $10 = 8 + 2$, $22 = 12 + 10$, $42 = 24 + 18$, and $86 = 44 + 42$.

Arranging the rows that are linearly dependent on each other in increasing order:

```
                +2  -4
                +2  -8
            -6  +8 +10 -12
           -10 +16 +18 -24
+22 -24 -26 +28 -38 +40 +42 -44
```

Looking at the differences between the rows:

```
              2
              6
          2   2   2
          6   2   6
  2   2   2  10   2   2   2
```

FIGURE 4

5. UNEXPLORED IDEAS AND UNUSED DATA

A necessary and sufficient condition for a matrix to have a zero determinant is that it have at least one eigenvalue that is zero. Unfortunately, deciding whether or not a matrix has a zero eigenvalue is no easier than deciding if it has a zero determinant. The only method not requiring direct calculation of the determinant involves finding linear subgraphs [5].

Figure 5 summarizes what we have discovered about the number of linear subgraphs of various types for the first few Pascal graphs. The number of linear subgraphs of $PG(n)$ grows very rapidly as n increases, limiting our pursuit of additional data. We have not yet discovered a pattern in these data that would point to a proof showing those Pascal matrices that have 0 determinants and those that have determinant 2.

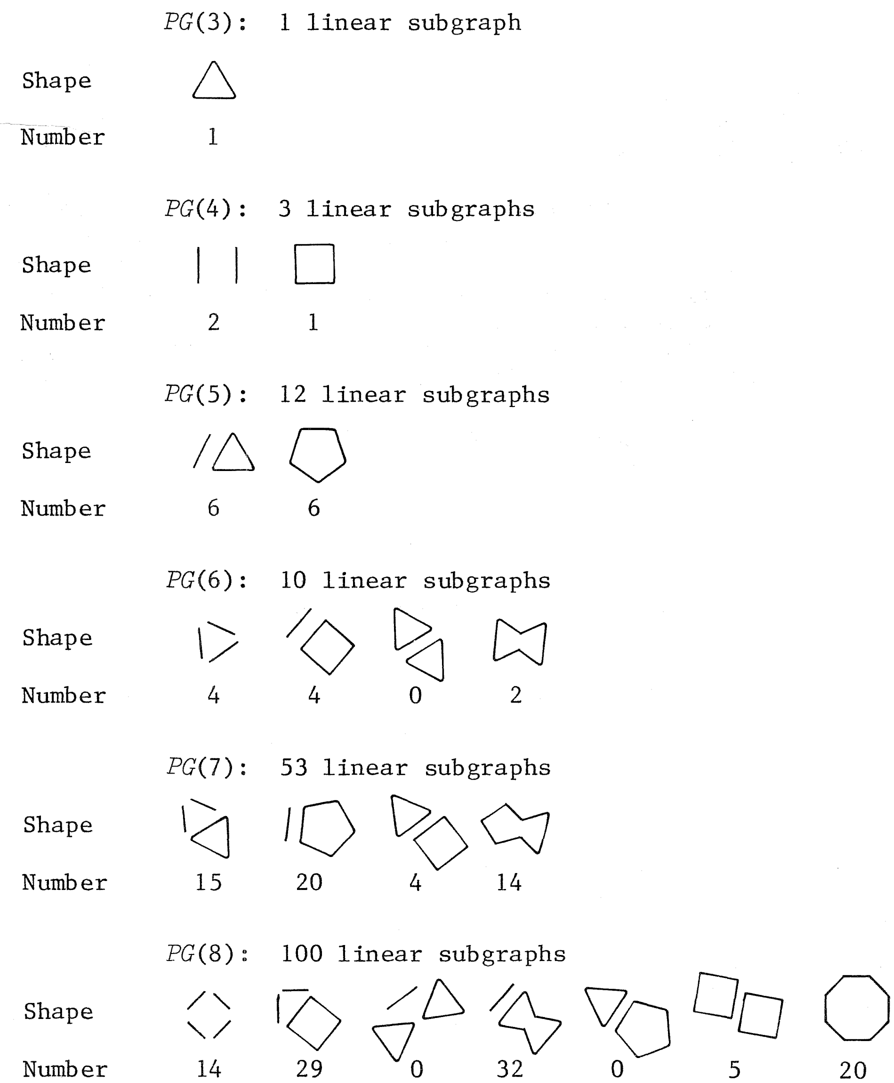


FIGURE 5

A topic that we have not explored is the eigenvalue spectra of the Pascal matrices. Since the matrices are symmetric, their eigenvalues are real. Perhaps a pattern in these spectra could be found. Several facts concerning the eigenvalue spectra may be useful. Let λ_i be one of the n eigenvalues of $PM(n)$; $\bar{d}_n$ be the mean valence of the vertices in $PM(n)$; r_n be the greatest eigenvalue of $PM(n)$. Then the number of edges in $PG(n)$ is

$$\sum_{i=1}^{n} \lambda^2/2;$$

the number of triangles in $PG(n)$ is

$$\sum_{i=1}^{n} \lambda^3/2;$$

and $\bar{d}_n \leqslant r_n \leqslant n - 1$ [2].

Table 1 lists the number of edges in the Pascal graphs of small order and Table 2 shows the vertex valency spectra of Pascal graphs of small order.

TABLE 1

n	Edges in $PG(n)$	n	Edges in $PG(n)$	n	Edges in $PG(n)$
1	0	8	19	15	57
2	1	9	27	16	65
3	3	10	29	17	81
4	5	11	33	18	83
5	9	12	37	19	87
6	11	13	45	20	91
7	15	14	49		

TABLE 2

n	Valency Spectrum for $PG(n)$																			
2	1	1																		
3	2	2	2																	
4	2	2	3	3																
5	3	3	4	4	4															
6	2	3	3	4	5	5														
7	3	3	4	4	4	6	6													
8	3	3	4	4	5	5	7	7												
9	4	4	5	5	6	6	8	8	8											
10	2	4	4	5	5	6	6	8	9	9										
11	3	4	4	4	5	6	6	6	8	10	10									
12	3	4	4	4	5	5	6	6	7	8	11	11								
13	4	4	5	5	5	6	6	7	8	8	8	12	12							
14	4	4	4	5	5	5	6	6	7	8	9	9	13	13						
15	5	5	5	5	5	5	6	6	8	8	8	10	10	14	14					
16	5	5	5	5	5	5	7	7	8	8	9	9	11	11	15	15				
17	6	6	6	6	6	6	8	8	9	9	10	10	12	12	16	16	16			
18	2	6	6	6	6	6	6	8	8	9	9	10	10	12	12	16	17	17		
19	3	4	6	6	6	6	6	6	8	8	9	10	10	10	12	12	16	18	18	
20	3	4	5	6	6	6	6	6	6	8	8	9	10	10	11	12	12	16	19	19

ACKNOWLEDGMENTS

The authors gratefully acknowledge the contributions of Professors Calvin Long and George Marsaglia through helpful and stimulating discussions.

REFERENCES

1. F. T. Boesch & A. P. Felzer. "A General Class of Invulnerable Graphs." In *Large Scale Networks: Theory and Design*. New York: IEEE Press, 1976.
2. D. Cvetkovic; M. Doob; & H. Sachs. *Spectra of Graphs: Theory and Application*. New York: Academic Press, 1980.
3. N. Deo. *Graph Theory with Applications to Engineering and Computer Science*. Englewood Cliffs, N.J.: Prentice-Hall, 1974.
4. N. J. Fine. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* 54 (1947):589-92.
5. F. Harary. "The Determinant of the Adjacency Matrix of a Graph." *SIAM Review* 4 (1962):202-10.
6. F. Harary. *Graph Theory*. Reading, Mass.: Addison-Wesley, 1969.
7. C. T. Long. "Pascal's Triangle Modulo p." *The Fibonacci Quarterly* 19 (1981):458-63.
8. W. F. Lunnon. "The Pascal Matrix." *The Fibonacci Quarterly* 15 (1977): 201-04.
9. V. A. Uspenskii. *Pascal's Triangle*. Trans. and adapted from the Russian by D. J. Sookne & T. McLarnan. Chicago: University of Chicago Press, 1974.

♦◊♦◊♦