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EQUIPROBABILITY IN THE FIBONACCI SEQUENCE

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For any positive integer m, the Fibonacci sequence is clearly periodic modulo m. Many moduli m, characterized in [1], have the property that *every* residue modulo m occurs in each period. (Indeed, 8 and 11 are the smallest moduli which do *not* have this property.) However, moduli m with the property that all m residues modulo m appear in one period *the same number of times* occur very infrequently, as the following theorem from [2] shows.

Theorem 1

If all m residues appear in one period of the Fibonacci sequence modulo m the same number of times, then m is a power of 5.

The converse of this theorem is also true [3]. Since (see [4]) for $k > 0$ the Fibonacci sequence modulo 5^k has period $4 \cdot 5^k$, it follows that if $m > 1$ is a power of 5, and (u_n) is the Fibonacci sequence, then every residue modulo m appears exactly four times in each sequence

$$u_s,\ u_{s+1},\ u_{s+2},\ \ldots,\ u_{s+4m-1}.$$

This result can be strengthened considerably.

Theorem 2

Denote the Fibonacci sequence by (u_n). If $m > 1$ is a power of 5, then every residue modulo m appears exactly once in each sequence

$$u_s,\ u_{s+4},\ u_{s+8},\ \ldots,\ u_{s+4(m-1)}.$$

Proof: Write $m = 5^k$, and denote the greatest integer function by [ ]. The Fibonacci sequence $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, ... satisfies the well-known formula

$$u_n = \big(((1 + \sqrt{5})2^{-1})^n - ((1 - \sqrt{5})2^{-1})^n\big)/\sqrt{5}.$$

Apply the binomial expansion to this formula to obtain

$$u_n = (2^{-1})^{n-1}\left(\binom{n}{1} + \binom{n}{3}5 + \binom{n}{5}5^2 + \cdots\right),$$

where all terms after $\binom{n}{2\ell + 1}5^{\ell}$ vanish and $\ell = [(n - 1)/2]$. Fix s, and let

$S_k = \{0, 1, \ldots, 5^k - 1\}$. Then, for $n = s + 4a$, $a \in S_k$, we have

$$u_n = (2^{-1})^{s-1}(2^{-1})^{4a}\left(\binom{n}{1} + \binom{n}{3}5 + \cdots\right)$$

and it is obvious that u_n represents every residue modulo 5^k if and only if

$$t_n = (2^{-1})^{4a}\left(\binom{n}{1} + \binom{n}{3}5 + \cdots\right)$$

represents every residue modulo 5^k, since s is fixed and $(2^{-1})^{s-1}$ is a unit modulo 5^k. Thus, we shall only consider t_n and prove the theorem by induction on k.

If $k = 1$, then $a \in \{0, 1, 2, 3, 4\}$ and $t_n \equiv s + 4a \pmod 5$, since $2^{-4} \equiv 1 \pmod 5$. Thus, the theorem is true for $k = 1$. Assume the theorem is true for k, and consider $k + 1$. For $a \in S_{k+1}$, write $a = b + c5^k$, where $b \in S_k$ and $c \in \{0, 1, 2, 3, 4\}$. Then,

$$\begin{aligned} t_n &= (2^{-1})^{4b}(2^{-1})^{4c5^k}\left(\binom{s + 4b + 4c5^k}{1} + \binom{s + 4b + 4c5^k}{3}5 + \cdots\right) \\ &\equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) + (2^{-1})^{4b}4c5^k \pmod{5^{k+1}}, \end{aligned}$$

since

$$(2^{-1})^{4\cdot 5^k} \equiv 1 \pmod{5^{k+1}}$$

and

$$\binom{s + 4b + 4c5^k}{2j + 1}5^j \equiv \binom{s + 4b}{2j + 1}5^j \pmod{5^{k+1}} \text{ for } j \geqslant 1.$$

[To prove the last congruence, note first that it is equivalent to

$$\binom{s + 4b + 4c5^k}{2j + 1}5^{j-1} \equiv \binom{s + 4b}{2j + 1}5^{j-1} \pmod{5^k}.$$

Then, observe that the power of 5 dividing $(2j + 1)!$ is exactly $j - 1$ for $j = 1, 2$, and is

$$\sum_{\ell=1}^{\infty} [(2j + 1)/5^\ell] \leqslant \sum_{\ell=1}^{\infty} (2j + 1)/5^\ell = (2j + 1)/4 \leqslant j - 1 \text{ for } j \geqslant 3.$$

Hence, $5^{j-1}/(2j + 1)!$ is integral at 5, and this implies the congruence.]

Let us now consider the congruence modulo 5^k. We obtain

$$t_n \equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) \pmod{5^k},$$

and, by the induction hypothesis, t_n represents the complete residue system modulo 5^k, for $n = s + 4b$, $b \in S_k$.

If we hold b fixed in S_k and let c run through the set $\{0, 1, 2, 3, 4\}$, we obtain

$$t_n \equiv (2^{-1})^{4b}\left(\binom{s + 4b}{1} + \binom{s + 4b}{3}5 + \cdots\right) + (2^{-1})^{4b}4c5^k \pmod{5^{k+1}},$$

which are all distinct residues modulo 5^{k+1} since $(2^{-1})^{4b}4c$ takes on distinct values modulo 5. Since the five t_n are all congruent to

$$(2^{-1})^{4b}\left(\binom{s+4b}{1} + \binom{s+4b}{3}5 + \cdots\right) \pmod{5^k},$$

the induction is complete.

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