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PROPERTIES OF SOME EXTENDED BERNOULLI AND EULER POLYNOMIALS

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1. INTRODUCTION

The study of Bernoulli, Euler, and Eulerian polynomials has contributed much to our knowledge of the theory of numbers. These polynomials are of basic importance in several parts of analysis and calculus of finite differences, and have applications in various fields such as statistics, numerical analysis, and so on. In recent years, the Eulerian numbers and certain generalizations have been found in a number of combinatorial problems (see [1], [3], [4], [5], [6], for example). A study of the above polynomials led us to the consideration of the following extension (3.1) of the Bernoulli, Euler, and Eulerian numbers, as well as polynomials in the unified form from a different point of view just described.

2. PRELIMINARY RESULTS

It is well known that the formulas [2]

$$g(n) = \sum_{d|n} f(d) \qquad (n = 1, 2, 3, \ldots) \tag{2.1}$$

and

$$f(n) = \sum_{cd=n} \mu(c)g(d) \qquad (n = 1, 2, 3, \ldots), \tag{2.2}$$

where $\mu(n)$ is the Mobius function, are equivalent. If in (2.1) and (2.2) we take $n = e_1e_2 \ldots e_r$, where the e_j are distinct primes, it is easily verified that (2.1) and (2.2) reduce to

$$g_r = \sum_{j=0}^{r} \binom{r}{j} f_j \qquad (r = 0, 1, 2, \ldots) \tag{2.3}$$

and

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g_j \qquad (r = 0, 1, 2, \ldots), \tag{2.4}$$

respectively, where for brevity we put

$$f_r = f(e_1e_2 \ldots e_r), \quad g_r = g(e_1e_2 \ldots e_r). \tag{2.5}$$

The equivalence of (2.3) and (2.4) is of course well known; the fact that the second equivalence is implied by the first is perhaps not quite so familiar. It should be emphasized that $f(n)$ and $g(n)$ are arbitrary arithmetic functions subject only to (2.1) or equivalently, (2.2); a like remark applies to f_r and g_r.

Given a sequence

$$f_r \qquad (r = 0, 1, 2, \ldots), \tag{2.6}$$

we define an extended sequence

$$f(n) \qquad (n = 1, 2, 3, \ldots) \tag{2.7}$$

such that

$$f(e_1 e_2 \ldots e_r) = f_r, \tag{2.8}$$

where the e_j are distinct primes. Clearly the extended sequence (2.7) is not uniquely determined by means of (2.8). If the sequence g_r is related to f_r by means of (2.3), then the sequence $g(n)$ defined by means of (2.1) furnishes an extension of the sequence g_r.

If we associate with the sequence f_r the (formal) power series

$$F_t = \sum_{r=0}^{\infty} f_r \frac{t^r}{r!}, \tag{2.9}$$

then (2.3) is equivalent to

$$G_t = \exp t \cdot F_t, \tag{2.10}$$

where

$$G_t = \sum_{r=0}^{\infty} g_r \frac{t^r}{r!}.$$

We associate with the sequence $f(n)$ the (formal) Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{2.11}$$

Then (2.1) is equivalent to

$$G(s) = \zeta(s)F(x), \tag{2.12}$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

3. EXTENDED POLYNOMIAL

We now define the extended polynomial set $B(n, h, a, k; x)$ using the following formula:

$$\frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^h - a} = \sum_{n=1}^{\infty} \frac{B(n, h, a, k; x)}{n^s}, \tag{3.1}$$

where a is a nonzero real number, k is a nonnegative integer, and $h \neq 0$.

On specializing various parameters involved therein we find the following relationships between our polynomials $B(n, h, a, k; x)$ and the extended Bernoulli, Euler, and other polynomials:

(i) Extended Bernoulli polynomials

$$B(n, h, 1, 1; x) = \beta(n, h; x) \tag{3.2}$$

(ii) Extended Bernoulli numbers

$$B(n, h, 1, 1; 0) = \beta(n, h) \tag{3.3}$$

(iii) Extended Euler polynomials

$$B(n, h, -1, 0; x) = \varepsilon(n, h; x) \tag{3.4}$$

(iv) Extended Euler numbers

$$B(n, h, -1, 0; 0) = \varepsilon(n, h) \tag{3.5}$$

(v) Extended Eulerian polynomials

$$B(n, h, a, 0; x) = \frac{2}{1 - a} H(n, h, a; x) \tag{3.6}$$

(vi) Extended Eulerian numbers

$$B(n, h, a, 0; 0) = \frac{a}{1 - a} H(n, h, a), \tag{3.7}$$

where the extended Bernoulli, Euler, and Eulerian polynomials and numbers are those introduced by Carlitz [2].

In the present paper we obtain numerous properties of the polynomials and numbers defined above. These properties are of an algebraic nature, and for the most part are generalizations on the corresponding properties of the Bernoulli, Euler, and Eulerian polynomials and numbers.

4. COMPLEMENTARY ARGUMENT THEOREM

Theorem 1

$$B(n, h, a, k; 1 - x) = \frac{(-1)^{k-1}}{a} B(n, -h, 1/a, k; x). \tag{4.1}$$

Proof: Consider the following:

$$\sum_{n=1}^{\infty}\frac{B(n, h, a, k; 1 - x)}{n^s} = \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{h(1-x)}}{(\zeta(s))^h - a}$$

$$= \frac{2\left(\frac{1}{2}h \log \zeta(x)\right)^k (\zeta(s))^{-hk}}{1 - a(\zeta(s))^{-h}}$$

$$= \frac{(-1)^{k-1}}{a}\cdot\frac{2\left(-\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{-hk}}{(\zeta(s))^{-h} - \frac{1}{a}}.$$

The theorem would follow if we interpret the above expression by (3.1).

Putting $x = 0$ in (4.1), we obtain

Corollary 1

$$B(n, h, a, k; 1) = \frac{(-1)^{k-1}}{a} B(n, -h, 1/a, k), \tag{4.2}$$

where (here and throughout this paper) $B(n, h, a, k; 0) = B(n, h, a, k)$.

5. RECURRENCE RELATIONS

To obtain some interesting results, we refer to [2] for the definition of $T_x(n)$:

$$(\zeta(s))^x = \sum_{n=1}^{\infty}\frac{T_x(n)}{n^s}, \tag{5.1}$$

where

$$T_x(n) = \prod_{e/h}\binom{j + x - 1}{j} \text{ with } n = \pi e^j,$$

and put

$$\log \zeta(s) = \sum_{n=1}^{\infty}\frac{\alpha(n)}{n^s}, \tag{5.2}$$

where

$$\alpha(n) = \begin{cases}\frac{1}{r} & (n = e^r,\ r \geqslant 1),\\ 0 & (\text{otherwise}).\end{cases} \tag{5.3}$$

We remark that $T_x(n)$ is a multiplicative function of n; that is,

$$T_x(mn) = T_x(m) \cdot T_x(n) \qquad [(m, n) = 1], \tag{5.4}$$

where (m, n) denotes the highest common divisor of two numbers m and n.

It is evident from (3.1) that

$$\sum_{n=1}^{\infty}\frac{B(n, h, a, k; x + y)}{n^s} = \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{h(x+y)}}{(\zeta(s))^h - a}$$

$$= \sum_{n=1}^{\infty} \frac{T_{hx}(n)}{n^s} \sum_{n=1}^{\infty} \frac{B(n, h, a, k; y)}{n^s},$$

which yields

Corollary 2

$$B(n, h, a, k; x + y) = \sum_{cd=n} T_{hx}(c)B(d, h, a, k; y). \tag{5.5}$$

In particular,

$$B(n, h, a, k; x) = \sum_{cd=n} T_{hx}(c)B(d, h, a, k). \tag{5.6}$$

From (3.1), it is easy to deduce the result:

$$\frac{d}{dx} B(n, h, a, k; x) = h \sum_{cd=n} \alpha(c)B(d, h, a, k; x). \tag{5.7}$$

Again, we may write (3.1) as

$$\sum_{n=1}^{\infty} \frac{B(n, 2h, a^2, k; x)}{n^s} = \frac{2(h \log \zeta(s))^k (\zeta(s))^{2hx}}{(\zeta(s))^{2h} - a^2}$$

$$= 2^{k-1} \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^h - a} \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^{k-1} (\zeta(s))^{hx}}{(\zeta(s))^h + a}$$

which gives a recurrence relation:

$$B(n, 2h, a^2, k; x) = 2^{k-1} \sum_{cd=n} B(c, h, a, 1; x)B(d, h, -a, k - 1; x). \tag{5.8}$$

Let us now consider the identity

$$\frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx/2}}{(\zeta(s))^{h/2} + a} = \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{h(x+1)/2}}{(\zeta(s))^h - a^2} - a \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx/2}}{(\zeta(s))^h - a^2}.$$

Because of the generating relation (3.1), we obtain:

$$2^k B(n, h/2, -a, k; x) = B\left(n, h, a^2, k; \frac{x + 1}{2}\right) - aB(n, h, a^2, k; x/2). \tag{5.9}$$

It follows from (3.1) that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}(B(n, h, a, k; x + 1) - aB(n, h, a, k; x)) = 2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{hx},$$

which implies:

$$B(n,\ h,\ a,\ k;\ x+1) - aB(n,\ h,\ a,\ k;\ x) = \frac{h^k}{2^{k-1}} \sum_{cd=n} \alpha_k(c) T_{hx}(d). \tag{5.10}$$

This leads to the summation formula:

$$\frac{h^k}{2^{k-1}} \sum_{cd=n} \alpha_k(c) \sum_{j=0}^{m-1} \frac{1}{a^j} T_{h(x+y)}(d)$$
$$= B(n,\ h,\ a,\ k;\ x+m) - aB(n,\ h,\ a,\ k;\ x). \tag{5.11}$$

It is easily verified that when $h = 1$ and $n = e_1 e_2 \ldots e_r$, (5.11) reduces to the familiar formula

$$D_n(x+m;\ a,\ k) - aD_n(x;\ a,\ k) = \frac{(n)_k}{2^{k-1}} \sum_{j=0}^{m-1} (x+j)^{n-k},$$

where

$(n)_k = n(n-1) \cdots (n-k+1)$ and $D_n(x;\ a,\ k)$ is defined in [7].

6. ADDITION THEOREMS

It may be of interest to deduce some addition theorems that are satisfied by $B(n,\ h,\ a,\ k;\ x)$.

Since

$$\frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{2hx}}{(\zeta(s))^h - a} \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{2hy}}{(\zeta(s))^h + a}$$
$$= 2 \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^{2k} (\zeta(s))^{2h(x+y)}}{(\zeta(s))^{2h} - a^2},$$

there follows at once:

Theorem 2

$$2^{2k-1} \sum_{cd=n} B(c,\ h,\ a,\ k;\ 2x) B(d,\ h,\ -a,\ k;\ 2y)$$
$$= B(n,\ 2h,\ a^2,\ 2k;\ x+y). \tag{6.1}$$

If we note the identity,

$$\frac{1}{(\zeta(s))^h - a} - \frac{1}{(\zeta(s))^h + a} = \frac{2a}{(\zeta(s))^{2h} - a^2}$$

then, as a consequence of (3.1), we arrive at:

Theorem 3

$$B(n,\ h,\ a,\ k;\ x) - B(n,\ h,\ -a,\ k,\ x) = \frac{a}{2^{k-1}}B(n,\ 2h,\ a^2,\ k;\ x/2). \tag{6.2}$$

On the other hand, since

$$\frac{1}{(\zeta(s))^h - a} - \frac{1}{(\zeta(s))^h - 1 + a} = \frac{2a - 1}{((\zeta(s))^h - a)((\zeta(s))^h - 1 + a)},$$

we get, from (3.1):

Theorem 4

$$\left(a - \frac{1}{2}\right)\sum_{cd=n} B(c,\ h,\ a,\ k;\ x)B(d,\ h,\ 1 - a,\ k;\ y)$$
$$= B(n,\ h,\ a,\ 2k;\ x + y) - B(n,\ h,\ 1 - a,\ 2k;\ x + y). \tag{6.3}$$

7. MULTIPLICATION THEOREMS

We establish the following multiplication theorems, in which m stands for a positive integer.

Theorem 5

$$\sum_{r=0}^{m-1} \frac{1}{a^r} B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right) = \frac{m}{a^{m-1}} B(n,\ h,\ a,\ k;\ mx). \tag{7.1}$$

Proof: In order to obtain (7.1), we have, from (3.1), the relation,

$$\sum_{n=1}^{\infty} \frac{B(n,\ h,\ a,\ k;\ mx)}{n^s} = \frac{2\left(\frac{1}{2}\ h \log \zeta(s)\right)^k (\zeta(s))^{mhx}}{(\zeta(s))^h - a}$$
$$= a^{m-1} \cdot \frac{2\left(\frac{1}{2} h \log \zeta(s)\right)^k (\zeta(s))^{mhx}}{(\zeta(s))^{mh} - a^m} \cdot \sum_{r=0}^{m-1} \frac{(\zeta(s))^{rh}}{a^r}$$
$$= \frac{a^{m-1}}{m^k} \sum_{r=0}^{m-1} \frac{1}{a^r} \sum_{n=1}^{\infty} \frac{B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right)}{n^s},$$

which completes the proof.

Proceeding exactly as in the proof of Theorem 5, and recalling (3.1), we obtain:

Theorem 6

$$\sum_{r=0}^{m-1} \frac{1}{a^r} B\left(n,\ mh,\ a^m,\ k;\ x + \frac{r}{m}\right) = \frac{m^k h}{2a^{m-1}} \sum_{cd=n} (c) B(d,\ h,\ a,\ k-1;\ mx). \tag{7.2}$$

Theorem 7

$$a^m p^k \sum_{r=0}^{m-1} \frac{1}{a^{rp}} B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)$$
$$= a^p m^k \sum_{q=0}^{p-1} \frac{1}{a^{qm}} B\left(n,\ ph,\ a^m,\ k;\ \frac{x}{p} + \frac{qm}{p}\right). \tag{7.3}$$

Proof: From (3.1), we have:

$$\sum_{n=1}^{\infty} \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)}{n^s}$$
$$= \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{2\left(\frac{1}{2}mh \log \zeta(s)\right)^k (\zeta(s))^{\left(\frac{x}{m} + \frac{rp}{m}\right)mh}}{(\zeta(s))^{mh} - a^m}$$
$$= m^k \cdot 2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx} \sum_{r=0}^{m-1} \frac{(\zeta(s))^{rph}}{a^{rp}}$$
$$= \frac{m}{a^{p(m-1)}} \cdot \frac{2\left(\frac{1}{2}h \log \zeta(s)\right)^k (\zeta(s))^{hx}}{(\zeta(s))^{mh} - a^m} \cdot \frac{(\zeta(s))^{mph} - a^{mp}}{(\zeta(s))^{ph} - a^p}. \tag{7.4}$$

From (7.4), and using symmetry in m and p, we obtain the required result.

Theorem 8

$$\sum_{r=0}^{m-1} \frac{1}{a^{rp}} B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)$$
$$= \frac{m^k h \cdot a^{p-m}}{2p^{k-1}} \sum_{q=0}^{p-1} \frac{1}{a^{qm}} \cdot \sum_{cd} \alpha(c) B\left(d,\ ph,\ a^m,\ k-1;\ \frac{x}{p} + \frac{qm}{p}\right). \tag{7.5}$$

Proof: It follows from (3.1) that

$$\sum_{n=1}^{\infty} \sum_{r=0}^{m-1} \frac{1}{a^{rp}} \cdot \frac{B\left(n,\ mh,\ a^m,\ k;\ \frac{x}{m} + \frac{rp}{m}\right)}{n^s}$$
$$= \frac{m^k h(\log \zeta(s))}{2p^{k-1}a^{p(m-1)}} \cdot \frac{2\left(\frac{1}{2}ph \log \zeta(s)\right)^{k-1} (\zeta(s))^{hx}}{(\zeta(s))^{ph} - a^p} \cdot \frac{(\zeta(s))^{mph} - a^{mp}}{(\zeta(s))^{ph} - a^p}$$
$$= \frac{m\ h \cdot a^{m(p-1)}}{2p^{k-1}a^{p(m-1)}} \cdot \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \sum_{n=1}^{\infty} \sum_{q=0}^{p-1} \frac{1}{a^{qm}} \cdot \frac{B\left(n,\ ph,\ a^p,\ k-1;\ \frac{x}{p} + \frac{qm}{p}\right)}{n^s},$$

which completes the proof.

Theorems 7 and 8 are elegant generalizations of Theorems 5 and 6, respectively.

8. ANOTHER MULTIPLICATION THEOREM

If we define the function $\overline{B}(n, h, a, k; x)$ by means of

$$\overline{B}(n, h, a, k; x) = B(n, h, a, k; x) \qquad (0 \leqslant x < 1),$$

$$\overline{B}(n, h, a, k; x + 1) = a\overline{B}(n, h, a, k; x),$$

then it is easily seen that multiplication formula (7.1) also holds for the barred function.

In this section we obtain an interesting generalization of (7.1) suggested by a recent result of Mordell [9]. In extending some results of Mikolas [8], Mordell proves the following theorem:

Let $u_1(x), \ldots, u_p(x)$ denote functions of x of period 1 that satisfy the relations

$$\sum_{r=0}^{m-1} u_i\left(x + \frac{r}{m}\right) = C_i^{(m)} u_i(mx) \qquad (i = 1, \ldots, p), \tag{8.1}$$

where $C_i^{(m)}$ is independent of x, let $a_1, \ldots, a_p$ be positive integers that are prime in pairs. Then, if the integrals exist and $A = a_1 a_2 \ldots a_p$,

$$\begin{aligned}
&\int_0^A u_1(x/a_1)u_2(x/a_2) \ldots u_p(x/a_p)\,dx \\
&= A\int_0^1 u_1(Ax/a_1)u_2(Ax/a_2) \ldots u_p(Ax/a_p)\,dx \\
&= C_1^{(a_1)} C_2^{(a_2)} \ldots C_p^{(a_p)} \int_0^1 u_1(x)u_2(x) \ldots u_p(x)\,dx.
\end{aligned} \tag{8.2}$$

We prove:

Theorem 8

Let $p \geqslant 1$; $n_1, \ldots, n_1 \geqslant 1$; $a_1, \ldots, a_p$ be positive integers that are relatively prime in pairs; $A = a_1 a_2 \ldots a_p$. Then,

$$\sum_{r=0}^{mA-1} \frac{1}{a^r}\, \overline{B}\left(n_1, ma_1h, a^{ma_1}, k; x_1 + \frac{r}{ma_1}\right) \cdot \overline{B}\left(n_2, ma_2h, a^{ma_2}, k; x_2 + \frac{r}{ma_2}\right)$$

$$\cdot \ldots \cdot \overline{B}\left(n_p, ma_ph, a^{ma_p}, k; x_p + \frac{r}{ma_p}\right)$$

(continued)

$$= C\sum_{r=0}^{m-1}\frac{1}{a^r}\overline{B}\left(n_1,\ mh,\ a^m,\ k;\ a_1x_1+\frac{r}{m}\right)\cdot\overline{B}\left(n_2,\ mh,\ a^m,\ k;\ a_2x_2+\frac{r}{m}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a\ x\ +\frac{r}{m}\right), \tag{8.3}$$

where

$$C = \frac{a_1^k a_2^k \ldots a_p^k}{a^{m(a_1-1)}a^{m(a_2-1)}\ \ldots\ a^{m(a_p-1)}}. \tag{8.4}$$

Proof: From (7.1), it is not difficult to show for arbitrary $a^* \geqslant 1$ that

$$\sum_{r=0}^{ma^*-1}\frac{1}{a}\overline{B}\left(n,\ ma^*h,\ a^{ma^*},\ k;\ x+\frac{r}{ma^*}\right)$$

$$=\sum_{r=0}^{m-1}\sum_{s=0}^{a^*-1}\frac{1}{a^{r+sm}}\overline{B}\left(n,\ ma^*h,\ a^{ma^*},\ k;\ x+\frac{s}{a^*}+\frac{r}{ma^*}\right)$$

$$=\frac{(a^*)^k}{a^{m(a^*-1)}}\cdot\sum_{r=0}^{m-1}\frac{1}{a^r}\overline{B}\left(n,\ mh,\ a^m,\ k;\ a^*x+\frac{r}{m}\right),$$

which agrees with (8.3) for $p = 1$.

For the general case, let S denote the left member of (8.3), and

$$A_s = a_1a_2\ \ldots\ a_s \qquad (1 \leqslant s \leqslant p).$$

If we replace r by $smA_{p-1} + r$, we have

$$S = \sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1+\frac{r}{ma_1}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1}+\frac{r}{ma_{p-1}}\right)$$

$$\cdot\sum_{s=0}^{a_{p-1}}\frac{1}{a^{sm}}\overline{B}\left(n_p,\ ma_ph,\ a^{ma_p},\ k;\ x_p+\frac{A_{p-1}s}{a_p}+\frac{r}{ma_p}\right)$$

$$=\sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1+\frac{r}{ma_1}\right)$$

$$\cdot\ \cdots\ \cdot\overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1}+\frac{r}{ma_{p-1}}\right)$$

$$\cdot\sum_{s=0}^{a_p-1}\frac{1}{a^{sm}}\overline{B}\left(n_p,\ ma_ph,\ a^{ma_p},\ k;\ x_p+\frac{s}{a_p}+\frac{r}{ma_p}\right)$$

$$=\frac{a_p^k}{a^{m(a_p-1)}}\sum_{r=0}^{mA_{p-1}-1}\frac{1}{a^r}\overline{B}\left(n_1,\ ma_1h,\ a^{ma_1},\ k;\ x_1+\frac{r}{ma_1}\right)$$

(continued)

$$\cdot \cdots \cdot \overline{B}\left(n_{p-1},\ ma_{p-1}h,\ a^{ma_{p-1}},\ k;\ x_{p-1} + \frac{r}{ma_{p-1}}\right)$$

$$\cdot \overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a_p x_p + \frac{r}{m}\right).$$

Continuing the same process, we get:

$$S = \frac{a_1^k a_2^k \ldots a_p^k}{a^{m(a_1-1)} a^{m(a_2-1)} \ldots a^{m(a_p-1)}}$$

$$\cdot \sum_{r=0}^{m-1} \frac{1}{a^r} \overline{B}\left(n_1,\ mh,\ a^m,\ k;\ a_1x_1 + \frac{r}{m}\right) \cdot \overline{B}\left(n_2,\ mh,\ a^m,\ k;\ a_2x_2 + \frac{r}{m}\right)$$

$$\cdot \cdots \cdot \overline{B}\left(n_p,\ mh,\ a^m,\ k;\ a_p x_p + \frac{r}{m}\right),$$

which completes the proof.

We remark that for $m = 1$, (8.3) reduces to

$$\sum_{r=0}^{A-1} \frac{1}{a^r} \overline{B}\left(n_1,\ a_1h,\ a^{a_1},\ k;\ x_1 + \frac{r}{a_1}\right) \cdot \cdots \cdot \overline{B}\left(n_p,\ a_ph,\ a^{a_p},\ k;\ x_p + \frac{r}{a_p}\right)$$

$$= C^*\overline{B}(n_1,\ h,\ a,\ k;\ a_1x_1) \cdot \cdots \cdot \overline{B}(n_p,\ h,\ a,\ k;\ a_px_p), \tag{8.5}$$

where

$$C^* = \frac{a_1^k \ldots a_p^k}{a^{a_1-1} \ldots a^{a_p-1}}.$$

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COMMENT ON PROBLEM H-315

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Recently I came to know Problem H-315 of *The Fibonacci Quarterly* (Vol. 18, 1980) which deals with "Kerner's method" for the simultaneous determination of polynomial roots. I want to comment on two aspects of the problem and its solution.

1. The method was already used by K. Wierstrass for a constructive proof of the fundamental theorem of algebra (cf. [1]). Kerner [2] realized that the method can be interpreted as a Newton method for the elementary symmetric functions; this fact is also observed in the textbook of Durand ([3], pp. 279-80) which appeared several years before Kerner's publication.

2. It is remarkable that the assumption

$$\sum_{i=1}^{n} z_i = -a_{n-1}$$

is *not* necessary for the validity of the assertion! This fact is mentioned by Byrnev and Dochev [4] where further references are given. The proof of the assertion

$$\sum_{i=1}^{n} \hat{z}_i = -a_{n-1}$$

is easy: following Kerner's derivation of the method, one must apply Newton's method to the system of elementary symmetric functions. Hence, one of the equations reads:

$$\sum_{i=1}^{n} x_i = -a_{n-1} \quad (x_1, x_2, \ldots, x_n \text{ denote the unknowns}).$$

[Please turn to page 188]