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# $n$-DIMENSIONAL FIBONACCI NUMBERS AND THEIR APPLICATIONS

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## INTRODUCTION

In one of his papers [3] Bernstein investigated the $F(n)$ function. This function was derived from a special kind of numbers which could well be defined as 3-dimensional Fibonacci numbers. The original Fibonacci numbers should then be called 2-dimensional Fibonacci numbers. The present paper deals with $n$-dimensional Fibonacci numbers in a sense to be explained in the sequel. In a later paper [4] Bernstein derived an interesting identity that was based on 3-dimensional Fibonacci numbers. Also Carlitz in his paper [5] deals with this subject.

If we remember that the original Fibonacci numbers are generated by the formula

$$F(n) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i}, \quad n = 1, 2, \ldots,$$

then the function

$$F(n) = \sum_{i=0} (-1)^i \binom{n-2i}{i}$$

can be regarded as a generalization of the first, and the author thought that

$$F(n) = \sum_{i=0} (-1)^i \binom{n-ki}{i}, \quad k = 1, 2, \ldots,$$

could serve as a $k$ - 1-dimensional generalization of the original Fibonacci numbers, but, regretfully, this consideration led nowhere. From the fact that the Fibonacci numbers are derived from the periodic expansion by the Euclidean algorithm of $\sqrt{5}$, there is opened a new horizon for the wanted generalization.

In a previous paper [1], the author had followed the ideas of Perron [9] and of Bernstein [4] and stated a general Algorithm that leads to an $n$-dimensional generalization of Fibonacci numbers.

In this paper, the author is introducing the GEA (Generalized Euclidean Algorithm) to investigate the various properties and applications of her $k$-dimensional Fibonacci numbers. It first turns out that these $k$-dimensional Fibonacci numbers are most useful for a good approximation of algebraic irrationals by rational integers. Further, the author proceeded to investigate higher-degree Diophantine equations and to state identities of a larger magnitude than those investigated before, in an explicit and simple form.

## 1. THE GEA

Let $w$ be the irrational

$$\begin{cases} w = \sqrt[n]{D^n + 1};\ n \geqslant 2,\ D \in N;\ x^{(v)} = (x_1^{(v)}(w),\ \ldots,\ x_{n-1}^{(v)}(w)), \\ \langle a^{(v)} \rangle,\ \langle b^{(v)} \rangle \text{ sequences of the form } x^{(v)},\ v = 0, 1, \ldots . \end{cases} \tag{1.1}$$

The GEA of the fixed vector $a^{(0)}$ is the sequence $\langle a^{(v)} \rangle$ obtained by the recurrency formula

$$\begin{cases} a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1}(a_2^{(v)} - b_2^{(v)},\ \ldots,\ a_{n-1}^{(v)} - b_{n-1}^{(v)},\ 1) \\ b_i^{(v)} = a_i^{(v)}(D);\ i = 1, \ldots, n-1;\ v = 0, 1, \ldots;\ a_1^{(v)} \neq b_1^{(v)}. \end{cases} \tag{1.2}$$

The GEA of $a^{(0)}$ is called purely periodic if there exists a number $m$ such that

$$\begin{cases} a^{(0)} = a^{(m)};\ m \text{ is called the length of the primitive period} \end{cases} \tag{1.3}$$

The following formulas were proved in [2]. Let

$$\begin{cases} A_s^{(v+n)} = \sum_{k=0}^{n-1} b_k^{(v)} A_s^{(v+k)};\ v = 0, 1, \ldots \\ A_i^{(j)} = \delta_i^j;\ \delta_i^j \text{ the Kronecker delta, } i, j = 0, 1, \ldots, n-1;\ s = 0, 1, \ldots, n-1; \\ b_k^{(v)} = a_k^{(v)}(D);\ k = 0, 1, \ldots, n-1;\ a_0^{(v)} = b_0^{(v)} = 1; \end{cases} \tag{1.4}$$

$A_s^{(v)}$ are called the matricians of GEA; then the three formulas hold:

$$\begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \ldots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \ldots & A_1^{(v+n-1)} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \ldots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)} \tag{1.5}$$

$$\begin{cases} a_s^{(0)} = \dfrac{\sum_{k=0}^{n-1} a_k^{(v)} A_s^{(v+k)}}{\sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)}},\ v = 0, 1, \ldots;\ s = 0, \ldots, n-1. \end{cases} \tag{1.6}$$

$$\prod_{k=1}^{v} a_{n-1}^{(k)} = \sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)} . \tag{1.7}$$

Perron proved the following theorem which, under the conditions of the GEA ($D \geqslant 1$), becomes

Theorem 1

The GEA is convergent in the sense that

$$\begin{cases} a_s^{(0)} = \dfrac{\lim\limits_{v \to \infty} A_0^{(v)}}{\lim\limits_{v \to \infty} A_0^{(v)}}, \quad s = 1, \ldots, n-1 \end{cases} \tag{1.8}$$

$A_s^{(v)} : A_0^{(v)}$ is called the $v$th convergent of GEA.

In [1], the author proved

Theorem 2

If the GEA of $a^{(0)}$ is purely periodic with $m =$ length of the primitive period, then

$$\begin{cases} \prod\limits_{k=0}^{m-1} a_{n-1}^{(k)} = \sum\limits_{k=0}^{n-1} a_k^{(m)} A_0^{(m+k)}, \\ \text{is a unit in } Q(w). \end{cases} \tag{1.9}$$

From (1.9) the formula follows, in virtue of (1.7),

$$\left(\prod_{k=0}^{m-1} a_{n-1}^{(k)}\right)^v = \sum_{k=0}^{n-1} a_k^{(m)} A_0^{(vm+k)}, \quad v = 1, 2, \ldots . \tag{1.10}$$

## 2. A PERIODIC GEA

In this section, we construct a periodic GEA, with length of primitive period $m = 1$. The fixed vector $a^{(0)}$ must be chosen accordingly, and this may look complicated at first. We prove

Theorem 3

The GEA of the fixed vector

$$\begin{cases} a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_s^{(0)}, \ldots, a_{n-1}^{(0)}) \\ a_s^{(0)} = \sum\limits_{i=0}^{s} \dbinom{n - s - 1 + i}{i} w^{s-i} D^i \\ s = 1, \ldots, n-1 \end{cases} \tag{2.1}$$

is purely periodic and the length of its primitive period $m = 1$.

Proof: We shall first need the formula

$$\sum_{i=0}^{s} \binom{n - s - 1 + i}{i} = \binom{n}{s}, \quad s = 1, \ldots, n-1. \tag{2.2}$$

This is proved by induction. The proof is left to the reader. We have, from (2.1), the following components of $a^{(0)}$ which we shall use later:

$$a_1^{(0)} = w + (n - 1)D; \; a_{n-1}^{(0)} = \sum_{i=0}^{n-1} w^{n-1-i}D^i. \tag{2.3}$$

Since $w^n - D^n = 1$, we also have

$$\sum_{i=0}^{n-1} w^{n-1-i}D^i = (w - D)^{-1}. \tag{2.4}$$

The vectors $b_i^{(v)}$ $(i = 1, \ldots, n - 1;\ v = 0, 1, \ldots)$ obtained from $a_i^{(v)}(w)$ by the defining rule (1.2) are called their corresponding companion vectors. We shall calculate the companion vector $b^{(0)}$ of $a^{(0)}$ and have

$$b_s^{(0)} = \sum_{i=0}^{s} \binom{n - s - 1 + i}{i} D^{s-i}D^i = D^s \sum_{i=0}^{s} \binom{n - s - 1 + i}{i},$$

so that, by (2.2),

$$b_s^{(0)} = \binom{n}{s} D^s, \; s = 1, 2, \ldots, n - 1. \tag{2.5}$$

Thus,

$$b^{(0)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \ldots, \binom{n}{n-1}D^{n-1}\right).$$

We shall now calculate the vector $a^{(1)}$. From (1.2), it follows that

$$a^{(1)} = (a_1^{(0)} - b_1^{(0)})^{-1}(a_2^{(0)} - b_2^{(0)}, \ldots, a_{n-1}^{(0)} - b_{n-1}^{(0)}, 1). \tag{2.6}$$

From (2.3), (2.4), and (2.5), we obtain:

$$\begin{cases} a_1^{(0)} - b_1^{(0)} = w + (n - 1)D - \binom{n}{1}D = w - D, \\ a_{n-1}^{(1)} = (w - D)^{-1} = \sum_{i=0}^{n-1} w^{n-1-i}D^i = a_{n-1}^{(0)}. \end{cases} \tag{2.7}$$

We can prove the relation

$$(a_s^{(0)} - b_s^{(0)})(a_1^{(0)} - b_1^{(0)})^{-1} = a_{s-1}^{(1)}, \; s = 2, \ldots, n - 1. \tag{2.8}$$

Since the proof is elementary, we leave it to the reader.

From (2.6), it follows that

$$\begin{cases} (a^{(1)} = a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-2}^{(0)}, a_{n-1}^{(0)}) = a^{(0)}, \\ a^{(v)} = a^{(0)}, \; v = 1, 2, \ldots . \end{cases} \tag{2.9}$$

This proves Theorem 3.

## 3. EXPLICIT MATRICIANS $A_0^{(v+n)}$

We shall proceed to find an explicit formula for the "zero-degree matricians" $A_0^{(v+n)}$, $v = 0, 1, \ldots$, and shall make use, for this purpose, of the defining formula (1.4), and the fact that the GEA is purely periodic with length of the primitive period $m = 1$. Taking into account (2.5) and (2.9), we have

$$A_0^{(v+n)} = \sum_{s=0}^{n-1} \binom{n}{s} D^s A_0^{(v+s)}; \quad v = 0, 1, \ldots . \tag{3.1}$$

We shall now make use of Euler's generating function. We have

$$\begin{aligned}
\sum_{i=0}^{\infty} A_0^{(i)} x^i &= x^0 A_0^{(0)} + \sum_{i=1}^{n-1} A_0^{(i)} x^i + \sum_{i=n}^{\infty} A_0^{(i)} x^i \\
&= 1 + \sum_{i=0}^{\infty} x^{i+n} \left( A_0^{(i)} + \binom{n}{1} D A_0^{(i+1)} + \binom{n}{2} D^2 A_0^{(i+2)} + \cdots + \binom{n}{n-1} D^{n-1} A_0^{i+n-1} \right) \\
&= 1 + x^n \sum_{i=0}^{\infty} A_0^{(i)} x^i + x^{n-1} \sum_{i=0}^{\infty} \binom{n}{1} D A_0^{(i+1)} x^{i+1} + x^{n-2} \sum_{i=0}^{\infty} \binom{n}{2} D^2 A_0^{(i+2)} x^{(i+2)} \\
&\quad + \cdots + x \sum_{i=0}^{\infty} \binom{n}{n-1} D^{n-1} A_0^{(i+n-1)} x^{i+n-1} \\
&= 1 + \left( x^n + \binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + \cdots + \binom{n}{n-1} D^{n-1} x \right) \sum_{i=0}^{\infty} A_i^{(0)} x^i \\
&\quad - \left( \binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + \cdots + \binom{n}{n-1} D^{n-1} x \right) = \sum_{i=0}^{\infty} A_0^{(i)} x^i,
\end{aligned}$$

$$\left[ 1 - \left( x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k} \right) \right] \sum_{i=0}^{\infty} A_0^{(i)} x^i = 1 - \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k},$$

$$\begin{aligned}
\sum_{i=0}^{\infty} A_0^{(i)} x^i &= \frac{1 - \left( x^n + \sum_{k=1}^{n} \binom{n}{k} D^k x^{n-k} \right) + x^n}{1 - \left( x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k} \right)} \\
&= \frac{x^n}{1 - \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k}} + 1,
\end{aligned}$$

$$A_0^{(0)} + \sum_{i=1}^{\infty} A_0^{(i)} x^i = x^n \sum_{t=0}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t + 1,$$

$$x A_0^{(1)} + A_0^{(2)} x^2 + \cdots + A_0^{(n-1)} x^{n-1} + \sum_{i=n}^{\infty} A_0^i x^i = x^n \sum_{t=0}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t,$$

For $x$ sufficiently small. Thus, since $A_0^{(1)} = \cdots = A_0^{(n-1)} = 0$, we have

$$\sum_{i=n}^{\infty} A_0^{(i)} x^i = x^n \sum_{t=0}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t,$$

$$\sum_{i=0}^{\infty} A_0^{(n+i)} x^{n+i} = x^n \sum_{t=0}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t,$$

$$\sum_{i=0}^{\infty} A_0^{(n+i)} x^i = \sum_{t=0}^{\infty} \left( \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t, \tag{3.2}$$

and comparing coefficients of powers $x^v$ on both sides of (3.2), we obtain

$$A_0^{(v+n)} = \sum_{ny_1 + (n-1)y_2 + \cdots + 2y_{n-1} + y_n = v} \binom{y_1 + y_2 + \cdots + y_n}{y_1, \ y_2, \ \ldots, \ y_n} \prod_{k=0}^{n-1} \left( \binom{n}{k} D^k \right)^{y_{k+1}} \tag{3.3}$$

or

$$A_0^{(v+n)} = \sum_{\sum_{i=0}^{n-1} (n-i) y_{i+1} = v} \binom{y_1 + y_2 + \cdots + y_n}{y_1, \ y_2, \ \ldots, \ y_n} D^{\sum_{j=1}^{n-1} j y_{j+1}} \prod_{k=0}^{n-1} \binom{n}{k}^{y_{k+1}}, \tag{3.4}$$

$$v = 0, 1, \ldots .$$

Formula (3.4) looks very complicated. $A_0^{(v+n)}$ can also be calculated by the recurrency relation (1.4). It is conjectured that it is easier to do so by formula (3.4), and would be a challenging computer problem.

## 4. MATRICIANS OF DEGREE $s$, $s = 1, 2, \ldots, n-1$

In this section, we shall express "$s$-degree matricians,"

$$A_s^{(v)}, \ s = 1, \ldots, n-1,$$

by means of zero-degree matricians. This is not an easy task. Now we shall prove a very important theorem.

### Theorem 4

The $s$-degree matricians are expressed through the zero-degree matricians by means of the relation

$$A_s^{(v+n-1)} = \sum_{k=0}^{s} \binom{n}{k} D^k A_0^{(v+n-s+k-1)}, \quad v = 0, 1, \ldots; \quad s = 1, \ldots, n-1. \tag{4.1}$$

Proof: From formula (1.6) it follows that

$$a_s^{(0)} \sum_{k=0}^{n-1} a_k^{(0)} A^{(v+k)} = \sum_{k=0}^{n-1} a_k^{(0)} A_0^{(v+k)}, \quad s = 1, 2, \ldots, n-1; \quad v = 0, 1, \ldots . \tag{4.2}$$

Or, writing $a_i$ for $a_i^{(0)}$, $i = 0, \ldots, n-1$, and substituting their values from (2.1), we obtain

$$\sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i \sum_{k=0}^{n-1} a_k A_0^{(v+k)} = \sum_{k=0}^{n-1} a_k A_s^{(v+k)}. \tag{4.3}$$

We shall now compare coefficients of $w^{n-1}$ on both sides of (4.3). The power of $w^{n-1}$ appears, on the right side only in

$$a_{n-1} = w^{n-1} + Dw^{n-2} + \cdots + D^{n-1},$$

and its coefficients is

$$A_s^{(v+n-1)}. \tag{4.4}$$

So the whole problem is to find the coefficient of $w^{n-1}$ on the left side, and this is *the* problem. We shall start with the first power of $w$ in $a_s$, which is $w^s$ (in the left side). Now in

$$\sum_{k=0}^{n-1} a_k A_0^{(n+k)}$$

we have to look for those $a_k$'s which have the powers $w^{n-s-1}$; this appears in

$$\begin{aligned}
&a_{n-s-1} \left(\text{first term, coefficient } = A_0^{(v+n-s-1)}\right)\\
&a_{n-s} \left(\text{second term, coefficient } = \binom{s}{1} DA_0^{(v+n-s)}\right)\\
&a_{n-s+1} \left(\text{third term, coefficient } = \binom{s}{2} D^2 A_0^{(v+n-s+1)}\right)\\
&\vdots\\
&a_{n-1} \left((1+s)\text{th term, coefficient } = \binom{s}{s} D^s A_0^{(v+n-1)}\right).
\end{aligned}$$

Thus, we have obtained the partial sum of coefficients of $w^{n-1}$ in the left side.

$$A_0^{(v+n-s-1)} + \binom{s}{1} DA_0^{(v+n-s)} + \binom{s}{2} D^2 A_0^{(v+n-s+1)} + \cdots + \binom{s}{s} D^s A_0^{(v+n-1)}.$$

Now the next power of $a_s$ on the left side is $w^{s-1}$ with coefficient

$$\binom{n-s-1+1}{1} D = \binom{n-s}{1} D.$$

To obtain $w^{n-1}$, $w^{s-1}$ must be multiplied by $n-s$, so we must start with the first term of $a_{n-s}$, the second term of $a_{n-s+1}$, ..., etc. Compared with the previous sum, $s$ has to be replaced by $s-1$. The sum will then be multiplied by $\binom{n-s}{1} D$, and the number of summands will be smaller by one. We then obtain the partial sum:

$$\binom{n-s}{1} D \left[ A_0^{(v+n-} + \binom{s-1}{1} DA_0^{(v+n-s+1)} + \binom{s-1}{2} D^2 A_0^{(v+n-s+2)} + \cdots + \binom{s-1}{s-1} D^{s-1} A_0^{(v+n-1)} \right].$$

Proceeding in this way, we obtained the partial sums:

$$
\left\{
\begin{aligned}
&A_0^{(v+n-s-1)} + \binom{s}{1}DA_0^{(v+n-s)} + \binom{s}{2}D^2A_0^{(v+n-s+1)} + \binom{s}{3}D^3A_0^{(v+n-s+2)} + \binom{s}{4}D^4A_0^{(v+n-s+3)} + \cdots + \binom{s}{k}D^kA_0^{(v+n-s-1+k)}\\
&\quad + \binom{n-s}{1}DA_0^{(v+n-s)} + \binom{n-s}{1}\binom{s-1}{1}D^2A_0^{(v+n-s+1)} + \binom{n-s}{1}\binom{s-1}{2}D^3A_0^{(v+n-s+2)} + \binom{n-s}{1}\binom{s-1}{3}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s}{1}\binom{s-1}{k-1}D^kA_0^{(v+n-s-1+k)}\\
&\qquad + \binom{n-s+1}{2}\binom{s-2}{0}D^2A_0^{(v+n-s+1)} + \binom{n-s+1}{2}\binom{s-2}{1}D^3A_0^{(v+s-n+2)} + \binom{n-s+1}{2}\binom{s-2}{2}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s+1}{2}\binom{s-2}{k-2}D^kA_0^{(v+n-s-1+k)}\\
&\qquad\quad + \binom{n-s+2}{3}\binom{s-3}{0}D^3A_0^{(v+n-s+2)} + \binom{n-s+2}{3}\binom{s-3}{1}D^4A_0^{(v+n-s+3)} + \cdots + \binom{n-s+2}{3}\binom{s-3}{k-3}D^kA_0^{(v+n-s-1+k)}\\
&\qquad\qquad + \cdots\cdots\cdots\cdots + \binom{n-s-1+k}{k}\binom{s-k}{k-k}D^kA_0^{(v+n-s-1+k)}\\
&\qquad\qquad\qquad + \cdots\cdots\cdots\cdots\\
&\text{(continued)}\\
&\quad + \cdots + \binom{s}{s-1}D^{s-1}A_0^{(v+n-2)} + \binom{s}{s}D^sA_0^{(v+n-1)}\\
&\quad + \cdots + \binom{n-s}{1}\binom{s-1}{s-2}D^{s-1}A_0^{(v+n-2)} + \binom{n-s}{1}\binom{s-1}{s-1}D^sA_0^{(v+n-1)}\\
&\quad + \cdots + \binom{n-s+1}{2}\binom{s-2}{s-3}D^{s-1}A_0^{(v+n-2)} + \binom{n-s+1}{2}\binom{s-2}{s-2}D^sA_0^{(v+n-1)}\\
&\quad + \cdots + \binom{n-s+2}{3}\binom{s-3}{s-4}D^{s-1}A_0^{(v+n-2)} + \binom{n-s+2}{3}\binom{s-3}{s-3}D^sA_0^{(v+n-1)}\\
&\quad + \cdots + \binom{n-s-1+k}{s-k-1}\binom{s-k}{s-k-1}D^{s-1}A_0^{(v+n-2)} + \binom{n-s-1+k}{k}\binom{s-k}{s-k}D^sA_0^{(v+n-1)}\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad + \binom{n-s-1+s}{s}\binom{s-s}{s-s}D^sA_0^{(v+n-1)}
\end{aligned}
\right.
\tag{4.5}
$$

Thus the general term in the sum of coefficients of $w^{n-1}$ on the left side of (4.3) which contains $D^k A_0^{(v+n-s-1+k)}$ as a constant factor has the form, adding up in (4.5) the column with this factor,

$$\sum_{i=0}^{k} \binom{n-s-1+i}{i}\binom{s-i}{k-i} D^k A_0^{(v+n-s-k+k)}. \tag{4.6}$$

The following formula is well known:

$$\sum_{i=0}^{k} \binom{n-s-1+i}{i}\binom{s-i}{k-i} = \binom{n}{k}, \tag{4.7}$$

which becomes formula (2.2) for $k = s$. Now, since in

$$a_s = \sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i,$$

the exponent of $D$ sums from $i = 0$ to $i = s$, we have, finally,

$$A_s^{(v+n-1)} = \sum_{k=0}^{s} \binom{n}{k} D^k A_0^{(v+n-1-s+k)}$$

which is formula (4.1) and proves Theorem 4. From formula (4.1), we have the single cases

$$A_1^{(v+n-1)} = A_0^{(v+n-2)} + \binom{n}{1} D A_0^{(v+n-1)}, \tag{4.8}$$

and

$$A_{n-1}^{(v+n-1)} = A_0^{(v+n)}. \tag{4.9}$$

(4.9) is a very surprising relation and will be applied in the next section. Similarly,

$$A_2^{(v+n-1)} = A_0^{(v+n-3)} + \binom{n}{1} D A_0^{(v+n-2)} + \binom{n}{2} D^2 A_0^{(v+n-s)}, \text{ etc.} \tag{4.10}$$

## 5. APPROXIMATION OF IRRATIONALS BY RATIONALS

We shall investigate especially the case $D = 1$, but produce first formulas for any value of $D$. We obtain from (4.8) and (1.6),

$$a_1^{(0)} = \frac{\lim_{v\to\infty} A_1^{(v+n-1)}}{\lim_{v\to\infty} A_0^{(v+n-1)}} = \lim_{v\to\infty} \frac{A_0^{(v+n-2)} + nDA_0^{(v+n-1)}}{A_0^{(v+n-1)}},$$

$$w + (n-1)D = nD + \lim_{v\to\infty} \frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}},$$

$$w = D + \lim_{v\to\infty} \frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}} = D + \lim_{v\to\infty} \frac{A_0^{(v+n-1)}}{A_0^{(v+n)}}. \tag{5.1}$$

For $D = 1$, $w = \sqrt[n]{2}$, and from (3.4) and (5.1) we obtain the approximation formula

$$\sqrt[n]{2} \approx \left\{ \frac{\displaystyle\sum_{\sum(n-i)y_{i+1}=v,\ i=0,\ldots,n-1} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}}{\displaystyle\sum_{\sum(n-i)y_{i+1}=v+1,\ i=0,\ldots,n-1} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}} \right.$$

$$b_k = \binom{n}{k},\ k = 0,\ \ldots,\ n-1;\ b_0 = 1.$$

The approximations are not very close, and we would have to continue a few steps further to get a closer approximation. Formula (4.9), surprisingly simple as it is, does not yield any news. It enables us to calculate $w^{n-1}$ by means of the powers $w_k$, $k = 1, \ldots, n-2$.

We have approximately, expanding $\sqrt[n]{2} = (1 + 1)^{1/n}$ by the binomial series,

$$\sqrt[n]{2} \approx 1 + \frac{1}{n}.$$

According to our approximation formula (5.1) with $D = 1$,

$$\sqrt[n]{2} = w \approx 1 + \frac{A_0^{(n)}}{A_0^{(n+1)}};$$

$$A_0^{(n+1)} = A_0^{(1)} + \binom{n}{1}A_0^{(2)} + \cdots + \binom{n}{n-1}A_0^{(n)} = \binom{n}{n-1}A_0^{(n)} = nA_0^{(n)} = n,$$

since $A_0^{(n)} = A_0^{(0)} + A_0^{(1)} + \cdots + \binom{n}{n-1}A_0^{(n-1)} = A_0^{(0)} = 1$, $\sqrt[n]{2} \approx 1 + \frac{1}{n}$, as should be.

## 6. DIOPHANTINE EQUATIONS

We shall construct two types of Diophantine equations of degree $n$ in $n$ unknowns and state their explicit solutions, which are infinite in number. We have from (1.5)

$$\begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \cdots & A_0^{(v+n+n-1)} \\ A_1^{(v+n)} & A_1^{(v+n+1)} & A_1^{(v+n+2)} & & A_1^{(v+n+n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v+n)} & A_{n-1}^{(v+n-1)} & A_{n-1}^{(v+n+2)} & & A_{n-1}^{(v+n+n-1)} \end{vmatrix} = (-1)^{(n-1)v}, \tag{6.1}$$

$$v = 0,\ 1,\ \ldots\ .$$

Substituting in (6.1) the values of $A_s^{(t)}$ from (4.1) we obtain, after simple row rearrangements,

$$\begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \cdots & A_0^{(v+n+n-1)} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n+1)} & \cdots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \cdots & A_0^{(v+n+n-3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_0^{(v+3)} & A_0^{(v+4)} & A_0^{(v+5)} & \cdots & A_0^{(v+n+2)} \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \cdots & A_0^{(v+n+1)} \\ A_0^{(v+1)} & A_0^{(v+2)} & A_0^{(v+3)} & \cdots & A_0^{(v+n)} \end{vmatrix} = (-1)^{(n-1)v} \tag{6.3}$$

We introduce the notations

$$X_{v,k} = A_0^{(v+k)}, \quad k = 1, 2, \ldots, n. \tag{6.3}$$

$$\begin{cases} A_0^{(v+k)} = A_0^{(v+k-n)} + b_1 A_0^{(v+k-n+1)} + b_2 A_0^{(v+k+2-n)} + \cdots + b_{n-1} A_0^{(v+k-1)} \\ b_k = \binom{n}{k} D^k, \quad k = 0, 1, \ldots, n-1, \quad v = 1, 2, \ldots . \end{cases} \tag{6.4}$$

We introduce these notations in (6.2) and then make the following manipulations in this determinant.

From the first row we subtract the $b_1$ multiple of the first row from below, then the $b_2$ multiple of the second row from below, ..., then the $b_k$th multiple of the $k$th row from below, $k = 1, \ldots, n - 1$.

Then (6.2) takes the form, in virtue of (6.4),

$$\begin{vmatrix} X_{v,n} - \sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & \cdots & X_{v,n-1} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n+1)} & \cdots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \cdots & A_0^{(v+n+n-3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \cdots & A_0^{(v+n+1)} \\ X_{v,1} & X_{v,2} & X_{v,3} & \cdots & X_{v,n} \end{vmatrix} = (-1)^{(n-1)v} \tag{6.5}$$

We further subtract from the second row the $b_2$ multiple of the first row from below, the $b_3$ multiple of the second row from below, ..., the $b_k$ multiple of the $(k - 1)$th row from below; the determinant (6.5) then takes the form ($k = 2, \ldots, n - 2$):

$$\left\{\begin{vmatrix} X_{v,n} - \sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & X_{v,3} & \cdots & X_{v,n-1} \\ X_{v,n-1} - \sum_{k=1}^{n-2} b_{k+1} X_{v,k} & X_{v,n} - \sum_{k=1}^{n-2} b_{k+1} X_{v,k+1} & X_{v,1} + b_1 X_{v,2} & X_{v,2} + b_1 X_{v,3} & \cdots & X_{v,n-2} + b_1 X_{v,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ V_{v,1} & X_{v,2} & X_{v,3} & X_{v,4} & \cdots & X_{v,n} \end{vmatrix}\right. \tag{6.6}$$
$$= (-1)^{(n-1)v}$$

Continuing this process by another step, the third row of determinant (6.6) will have the form

$$X_{v,n-2} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k} \quad X_{v,n-1} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k+1} \quad X_{v,n} - \sum_{k=1}^{n-3} b_{k+2} X_{v,k+2}$$

$$X_{v,1} + b_1 X_{v,2} + b_2 X_{v,3} \quad X_{v,2} + b_1 X_{v,3} + b_2 X_{v,4} \quad \cdots$$

$$X_{v,n-3} + b_1 X_{v,n-2} + b_2 X_{v,n-1}.$$

Generally we subtract from the th row in (6.2) the $b_i$ multiple of the first row from below, then the $b_{i+1}$ multiple of the second row from below, ..., the $b_{n-1}$ multiple of the $(n-i)$th row from below ($i = 1, \ldots, n-1$). The reader can verify, that by these operations the determinant (6.2) transforms into one containing only the unknowns $X_{v,i}$ ($i = 1, \ldots, n$), which yields the Diophantine equation of degree $n$ in these unknowns.

## 7. MORE DIOPHANTINE EQUATIONS

The GEA of $a^{(0)}$ is purely periodic with length of the primitive period $m = 1$. Since

$$a_{n-1}^{(0)} = \sum_{i=0}^{n-1} \binom{n-1-(n-1)+i}{i} w^{n-1-i} D^i = \sum_{i=0}^{n-1} w^{n-1-i} D^i$$

we have by Theorem 2 and formula (1.10),

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v = \sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)}, \quad v = 1, 2, \ldots . \tag{7.1}$$

We find the norm of $(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v$. We have

$$\begin{cases} D^n - w^n = -1, \\ D^n - w^n = -\sum_{k=0}^{n-1} (D - \rho_k w) = -N(D - w), \\ \rho_k = e^{2\pi i k/n}, \; k = 0, 1, \ldots, n-1. \end{cases} \tag{7.2}$$

But $w^{n-1} + Dw^{n-2} + \cdots + D^{n-1} = -(D - w)^{-1}$; hence,

$$N[(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v] = (-1)^{(n-1)v}, \quad v = 1, 2, \ldots; \quad n = 2, 3, \ldots . \tag{7.3}$$

We have

$$\sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)} = A_0^{(v)} + \left[w + \binom{n-1}{1} D\right] A_0^{(v+1)} + \left[w^2 + \binom{n+2}{1} Dw + \binom{n-1}{2} D^2\right] A_0^{(v+2)}$$
$$+ \left[w^3 + \binom{n-3}{1} w^2 D + \binom{n-2}{2} wD^2 + \binom{n-1}{3} D^3\right] A_0^{(v+3)} + \cdots$$
$$+ \left[w^{n-1} + \binom{1}{1} w^{n-1} D + \binom{2}{2} w^{n-2} D^2 + \cdots + \binom{n-1}{n-1} D^{n-1}\right] A_0^{(v+n-1)}.$$

Denoting

$$\begin{cases} X_{v,k} = \displaystyle\sum_{s=0}^{n-1-k} \binom{n-1-k}{s} A_0^{(v+s+k)} D^s, \\ k = 0, 1, \ldots, n-s. \end{cases} \tag{7.4}$$

This $X_{v,k}$ is not the $X_{v,k}$ from (6.4). We have from (7.1),

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^v = \sum_{k=0}^{n-1} X_{v,k} w^k = e^v, \quad e \text{ a unit.} \tag{7.5}$$

We shall find the field equation of

$$\sum_{k=0}^{n-1} X_{v,k} w^k.$$

The free member of it is the norm of $e^v$, and since $e^v$ is a unit with the norm $(-1)^{(n-1)v}$, according to (7.3), we find easily, by known methods, that

$$\begin{vmatrix} X_{v,0} & X_{v,1} & X_{v,2} & \cdots & X_{v,n-2} & X_{v,n-1} \\ mX_{v,n-1} & X_{v,0} & X_{v,1} & \cdots & X_{v,n-3} & X_{v,n-2} \\ mX_{v,n-2} & mX_{v,n-1} & X_{v,0} & \cdots & X_{v,n-4} & X_{v,n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ mX_{v,2} & mX_{v,3} & mX_{v,4} & \cdots & X_{v,0} & X_{v,1} \\ mX_{v,1} & mX_{v,2} & mX_{v,3} & \cdots & mX_{v,n-1} & X_{v,1} \end{vmatrix} = (-1)^{(n-1)v} \tag{7.6}$$

It is not difficult to see that, in the case $n = 2m + 1$ $(m = 1, 2, \ldots)$, the highest powers of the $n$ unknowns of the discriminant (7.6) as

$$X_{v,0}^n, \; mX_{v,1}^n, \; m^2 X_{v,2}^n, \; \ldots, \; m^{n-1} X_{v,n-1}^n,$$

while the last unknown, $X_{v,n-1}$ does not have the exponent $n$, but a smaller one. In the case $n = 2m$ ($m = 1, 2, \ldots$) these $n - 1$ powers are the same, but with alternating signs, viz.,

$$X_{v,0}^n, \ -mX_{v,1}^n, \ +m^2X_{v,2}^n, \ \ldots .$$

In the case $n = 2$, the expanded discriminat (7.6) had the form

$$X_v^2 - mY_v^2 = \pm 1,$$

and in the case $n = 3$, it had the form

$$X^3 + mY^3 + m^2Z^3 - 3mXYZ = 1.$$

The first is Pell's equation.

## 8. IDENTITIES AND UNITS

We return to formulas (7.4) and (7.5), and have

$$\begin{cases} X_{nv,k} = \displaystyle\sum_{s=0}^{n-1-k} \binom{n-1-k}{s} A_0^{(vn+s+k)} D^s \\ \\ k = 0, 1, \ldots, n-1 \\ \\ (w^{n-1} + Dw^{n-2} + \cdots + D^{n-1}) = \displaystyle\sum_{k=0}^{n-1} X_{nv,k}\, w^k. \end{cases} \tag{8.1}$$

We compare powers of $w^k$ ($k = 0, 1, \ldots, n-1$) on both sides of (8.1) and take into consideration that $w^{nt} = m^t = (D^n + 1)^t$. We have, looking for the rational part of the right side, $k = 0$, and the value of the right side equals $X_{nv,0}$, and by (7.4),

$$X_{nv,0} = \sum_{s=0}^{n-1} \binom{n-1}{s} A^{(nv+s)} D^s, \quad v = 0, 1, \ldots . \tag{8.2}$$

On the left side, we have to look for the coefficients of $w^n$. Since the highest power in the expression

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv}$$

is $n(n-1)v$, we have the expression

$$\sum_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn \leqslant n(n-1)v, \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v - sn,\ s = 0, 1, \ldots, (n-1)v}} \binom{y_1 + y_2 + \cdots + y_n}{y_1,\ y_2,\ \ldots,\ y_n} w^{\sum_{i=1}^{n-1}(n-i)y_i} D^{\sum_{i=1}^{n-1} iy_{i+1}} = X_{nv}, \tag{8.3}$$

We want to obtain in this way the rational part of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv}.$$

At the same time

$$\sum_{i=1}^{n-1} iy_{i+1}$$

is the sum of the exponents of the powers of $y_{i+1}$ ($i = 1, \ldots, n - 1$). Since in every summand of

$$w^{n-1} + Dw^{n-2} + \cdots + D^{n-1}$$

the sum of the exponents of $D^i w^{n-1-i}$ ($i = 0, 1, \ldots, n - 1$) is $n - 1$, and the highest exponent in the expansion if $n(n - 1)v$, we have that

$$\sum_{i=1}^{n-1} (n - i)y_i + \sum_{i=1}^{n-1} iy_{i+1} = n(n - 1)v,$$

which explains the left side of (8.3). We further have

$$\sum_{i=1}^{n-1} [(n - i)y_i + iy_{i+1}] = n(n - 1)v,$$

so that

$$y_1 + y_2 + \cdots + y_n = nv. \tag{8.4}$$

Now, taking into account that the exponent of $w$ under the summation sign in (8.3) equals $sn$, $w^{sn} = m^s$, and $D^n = m - 1$, formula (8.3) takes the form

$$\begin{cases} \displaystyle\sum_{\sum_{i=1}^{n-1}(n-i)y_i = sn} \binom{nv}{y_1, y_2, \ldots, y_n} m^s (m - 1)^{(n-1)v-s} = X_{nv,0} \\ \displaystyle = \sum_{k=0}^{n-1} \binom{n-1}{k} D^k A_0^{(nv+k)}, \\ s = 0, 1, \ldots, (n - 1)v;\ v = 0, 1, \ldots \\ y_1 + y_2 + \cdots + y_n = nv \end{cases} \tag{8.5}$$

(8.5) is an interesting combinatorial identity.

From (8.1), $n - 1$ more identities can be obtained by comparing the coefficients of the powers $w^i$, $i = 1, \ldots, n - 1$, on both sides of (8.1). The identities have a somewhat complicated form; however, they will express the coefficients of $w^t$, $t = 1, \ldots, n - 1$, in the expansion of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{nv},$$

with $w^n = m = D^n + 1$:

$$\begin{cases} \displaystyle\sum_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn+t \leqslant n(n-1)v \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v-(sn+t)}} \binom{nv}{y_1, y_2, \ldots, y_n} m^s(m-1)^{(n-1)v-s-1}D^{n-t} = X_{nv,t} \\ \displaystyle = \sum_{j=0}^{n-1-t}\binom{n-1-t}{j}A_0^{(nv+j+t)}D^j \\ j = 0, 1, \ldots, (n-1)v-1; \; t = 1, \ldots, n-1. \end{cases} \tag{8.6}$$

We wish to explain the appearance of the factor $D^{n-t}$ under the summation sign on the left side of (8.6). The power of $D$ in the expantion of

$$(w^{n-1} + Dw^{n-2} + \cdots + D^{n-1})^{vn}$$

equals

$$\begin{aligned}\sum_{i=1}^{n-1} iy_{i+1} &= n(n-1)v - (sn+t) \\ &= n(n-1)v - sn - n + (n-t) \\ &= n[(n-1)v - s - 1] + n - t.\end{aligned}$$

Thus, the power of $D$ equals

$$(D^n)^{n(n-1)v-s-1} \cdot D^{n-t}, \text{ with } D^n = m-1.$$

The power of $w$ is

$$\sum_{i=1}^{n-1}(n-i)y_i = sn + t = (w^n)^s w^t = m^s w^t,$$

so $m^s$ is the coefficient of $w^t$ as desired.

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