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A GENERAL METHOD FOR DETERMINING A CLOSED FORMULA FOR THE NUMBER OF PARTITIONS OF THE INTEGER n INTO m POSITIVE INTEGERS FOR SMALL VALUES OF m

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The function $p\ (n)$ is defined as the number of partitions of the integer n into exactly m nonzero positive integers where the order is irrelevant. A general method for determining a formula for $p_m(n)$ for small values of m is given. The formulas are simpler in form than any previously given.

1. INTRODUCTION

If $p_m(n)$ is the number of partitions of the integer n into exactly m positive integers and if $p_m^{\star}(n)$ is the number of partitions into at most m parts and $p(m)$ is the usual partition function, then there are some simple known relationships between them.

$$p_m(n) - p_m(n - m) = p_{m-1}(n - 1)$$
$$p_m^{\star}(n) = p_m(n + m)$$
$$p(m) = p_m(2m)$$

The first recurrence relationship can be solved sequentially starting with $m = 2$ to determine the exact solution for small values of m. The method is given in Section 2. The procedure is to determine the complementary function and the particular solution to satisfy the m initial conditions $p_m(n) = 0$ for $0 \leqslant n \leqslant m - 1$ starting with $p_1(n) = 1$. This leads to the following forms.

$$p_2(n) = \left[\frac{n}{2!1!}\right] \qquad p_3(n) = \left[\frac{n^2 + 3}{3!2!}\right]$$

$$p_4(n) = \left[\frac{n^3 + 3n^2 + \frac{1}{2}\{9n(-1)^n - 9n\} + 32}{4!3!}\right]$$

$$p_5(n) = \left[\frac{n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n + 905}{5!4!}\right]$$

$$p_6(n) = \left[\frac{n^5+22\frac{1}{2}n^4+126\frac{2}{3}n^3-112\frac{1}{2}n^2-1599\frac{1}{6}n+112\frac{1}{2}(-1)^n(n^2+9n)+1066\frac{2}{3}n\cos\frac{2n\pi}{3}+19224}{6!5!}\right].$$

A GENERAL METHOD FOR DETERMINING A CLOSED FORMULA

HISTORICAL NOTE

Exact determinations of $p_m(n)$ for small m have been given in a variety of forms by many writers. See Dickson's *History of the Theory of Numbers*, Vol. 2, and *The Royal Society Mathematical Tables*, Vol. 4, by H. Gupta and others for extensive details of previous work together with references. De Morgan (1843) gives formulas for $p_3(n)$ and $p_4(n)$ which are equivalent to the above forms (see Dickson, p. 115). In Gupta (p. xvi), formulas are quoted in the form below, where $p(n, m) = p_m(n + m)$.

$$p(n,\ 1) = 1$$

$$p(n,\ 2) = \frac{1}{2}\left(n + \frac{3}{2}\right) + \frac{1}{4}(-1)^n$$

$$p(n,\ 3) = \frac{1}{12}\left(n^2 + 6n + \frac{47}{6}\right) + \frac{1}{8}(-1)^n + \frac{1}{9}(\alpha_3^n + \alpha_3^{2n})$$

$$p(n,\ 4) = \frac{1}{144}\left(n^3 + 15n^2 + \frac{135n}{2} + \frac{175}{2}\right) + \frac{1}{32}(n+5)(-1)^n + \frac{i}{9\sqrt{3}}(\alpha_3^{n-1} - \alpha_3^{2n-2}) + \frac{1}{16}(i^n + i^{3n})$$

where $\alpha_3 = \exp\dfrac{2i\pi}{3}$ is a cube root of unity.

This development is essentially due to J. W. L. Glaisher (1908) (see Gupta and Dickson, p. 117). Glaisher obtained complete results to $m = 10$ and the results are given to $m = 12$ in Gupta, but the formulas obtained are very complicated.

Further results are given in Gupta, but all the exact formulas given for small m are more complicated than those given here.

SECTION 2

Write the recurrence equation in the form

$$p_m(n + m) - p_m(n) = p_{m-1}(n + m - 1).$$

The solution of this equation is composed of two parts.

1. The complementary function given by the solution of

$$p_m(n + m) - p_m(n) = 0.$$

This simply gives the form

$$a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_m\alpha_m^n,$$

where the a_i are constants and the α_i are the mth roots of unity where $\alpha_1 = 1$ (say).

2. The particular solution is determined apart from the arbitrary constant which is included in (1) by the solution of the equation

$$\Delta(m)\{p_m(n)\} = p_{m-1}(n + m - 1),$$

where $\Delta(m)$ is an operator such that

$$\Delta(m)\{p_m(n)\} = p_m(n + m) - p_m(n).$$

Thus we can write formally

$$p_m(n) = \frac{1}{\Delta(m)}\{p_{m-1}(n + m - 1)\},$$

where $\dfrac{1}{\Delta(m)}$ is the inverse operator to $\Delta(m)$.

To Determine the Action of $\frac{1}{\Delta(m)}$

2.1 Let $p(n)$ be any polynomial function in n with constant coefficients. Then

$$\frac{1}{\Delta(m)}\{p(n)\} = \left(\frac{1}{mD} + B_1 + \frac{B_2 mD}{2!} + \frac{B_4 m^3 D^3}{4!} + \frac{B_6 m^5 D^5}{6!} + \cdots\right)\{p(n)\},$$

where the B are the Bernoulli numbers and the right-hand side is finite as $p(n)$ is a polynomial. This is a well-known result.

2.2 Consider $\Delta(m)\left\{\dfrac{\alpha^n}{\alpha^m - 1}\right\}$, where $\alpha^m \neq 1$

$$= \frac{\alpha^{n+m} - \alpha^n}{\alpha^m - 1} = \alpha^n$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{\alpha^n}{\alpha^m - 1} \quad \text{when} \quad \alpha^m \neq 1.$$

2.3 Consider $\Delta(m)\left\{\dfrac{n\alpha^n}{m}\right\}$, where $\alpha^m = 1$

$$= \frac{(n + m)\alpha^{n+m} - n\alpha^n}{m} = \alpha^n$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{n\alpha^n}{m} \quad \text{when} \quad \alpha^m = 1.$$

2.4 Let $f(n)$ and $g(n)$ be any functions of n; then

$$\begin{aligned}\Delta(m)\{f(n)g(n)\} &= f(n + m)g(n + m) - f(n)g(n)\\ &= f(n + m)g(n + m) - f(n)g(n + m) + f(n)g(n + m)\\ &\qquad - f(n)g(n)\end{aligned}$$

(continued)

$$= g(n+m)\Delta(m)\{f(n)\} + f(n)\Delta(m)\{g(n)\}$$

$$\therefore \quad f(n)g(n) = \frac{1}{\Delta(m)}\Big\{g(n+m)\Delta(m)\{f(n)\}\Big\} + \frac{1}{\Delta(m)}\Big\{f(n)\Delta(m)\{g(n)\}\Big\}$$

$$\therefore \quad \frac{1}{\Delta(m)}\Big\{f(n)\Delta(m)\{g(n)\}\Big\} = f(n)g(n) - \frac{1}{\Delta(m)}\Big\{g(n+m)\Delta(m)\{f(n)\}\Big\}$$

Put $\Delta(m)\{g(n)\} = \alpha^n$.

$$\therefore \quad g(n) - \frac{1}{\Delta(m)}\{\alpha^n\} = \frac{\alpha^n}{\alpha^m - 1} \text{ if } \alpha^m \neq 1$$

$$\therefore \quad \frac{1}{\Delta(m)}\{f(n)\alpha^n\} = f(n) \cdot \frac{\alpha^n}{\alpha^m - 1} - \frac{1}{\Delta(m)}\left\{\frac{\alpha^{n+m}}{\alpha^m - 1}\Delta(m)\{f(n)\}\right\}$$

$$= \frac{f(n)\alpha^n}{\alpha^m - 1} - \frac{\alpha^m}{\alpha^m - 1}\,\frac{1}{\Delta(m)}\Big\{\alpha^n\Delta(m)\{f(n)\}\Big\}.$$

Thus, if $f(n)$ is a polynomial in n, then this is a reduction formula that can be successively applied to determine the left-hand side. From which it follows that if $\alpha^m \neq 1$ and $f(n)$ is a polynomial of degree p, we have

$$\frac{1}{\Delta(m)}\{\alpha^n f(n)\} = \frac{\alpha^n}{\alpha^m - 1}\left(1 + \frac{\alpha^m}{\alpha^m - 1}\Delta(m)\right)^{-1}\{f(n)\}$$

$$= \frac{\alpha^n}{\alpha^m - 1}\left(1 - \frac{\alpha^m}{\alpha^m - 1}\Delta + \left(\frac{\alpha^m}{\alpha^m - 1}\right)^2\Delta^2 - \cdots + (-1)^p\left(\frac{\alpha^m}{\alpha^m - 1}\right)^p\Delta^p\right)\{f(n)\}.$$

2.5 Consider $\Delta(m)\{f(n)\alpha^n\}$, where $\alpha^m = 1$.

$$\therefore \quad \Delta(m)\{f(n)\alpha^n\} = f(n+m)\alpha^{n+m} - f(n)\alpha^n$$

$$= \alpha^n(f(n+m) - f(n))$$

$$= \alpha^n\Delta(m)\{f(n)\}$$

$$\therefore \quad f(n)\alpha^n = \frac{1}{\Delta(m)}\Big\{\alpha^n\Delta(m)\{f(n)\}\Big\}.$$

Put $\Delta(m)\{f(n)\} = p(n)$.

$$\therefore \quad f(n) = \frac{1}{\Delta(m)}\{p(n)\}$$

$$\therefore \quad \frac{1}{\Delta(m)}\{\alpha^n p(n)\} = \alpha^n f(n) = \alpha^n\frac{1}{\Delta(m)}\{p(n)\} \text{ if } \alpha^m = 1.$$

Thus, if $p(n)$ is a polynomial and $\alpha^m = 1$, we have

$$\frac{1}{\Delta(m)}\{\alpha^n p(n)\} = \alpha^n\left(\frac{1}{mD} + B_1 + \frac{B_2 mD}{2!} + \frac{B_4 m^3 D^3}{4!} + \cdots\right)\{p(n)\}.$$

This determines the action of $\frac{1}{\Delta(m)}$ in all cases. Thus, for

$$p_m(n + m) - p_m(n) = p_{m-1}(n + m - 1),$$

we have that

$$p_m(n) = a_1\alpha_1^n + a_2\alpha_2^n + \cdots + a_m\alpha_m^m + \frac{1}{\Delta(m)}\{p_{m-1}(n + m - 1)\},$$

where the a_i are constants and the α_i are the mth roots of unity with $\alpha_1 = 1$. We have the m conditions $p_m(n) = 0$ for $0 \leqslant n \leqslant m - 1$ for the determination of the m constants.

Thus, the $p_m(n)$ can be determined sequentially for values of m starting with $m = 2$.

Now, $p_1(n) = 1$,

$$\therefore \quad p_2(n + 2) - p_2(n) = p_1(n + 1) = 1$$

$$\therefore \quad p_2(n) = a_1(1)^n + a_2(-1)^n + \frac{1}{\Delta(2)}\{1\}$$

$$= a_1 + a_2(-1)^n + \frac{n}{2}$$

Now, $p_2(0) = a_1 + a_2 = 0$

$$p_2(1) = a_1 - a_2 + \frac{1}{2} = 0$$

$$\therefore \quad a_1 = -\frac{1}{4}, \quad a_2 = \frac{1}{4}$$

$$\therefore \quad p_2(n) = -\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{n}{2}$$

Now $p_2(n)$ is an integer for all positive integral n. Now

$$\max\left\{-\frac{1}{4} + \frac{1}{4}(-1)^n\right\} = 0, \text{ for } n = 2 \text{ (say)}.$$

Thus, we can write $p_2(n) = \left[\frac{n}{2}\right]$.

$\underline{m = 3}$

$$\therefore \quad p_3(n + 3) - p_3(n) = p_2(n + 2) = -\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{n + 2}{2}$$

$$\therefore \quad p_3(n) = a_1 + a_2\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + a_3\left(\frac{-1 - i\sqrt{3}}{2}\right)^n - \frac{n}{12} = \frac{1}{8}(-1)^n + \frac{(n + 2)^2}{12} - \frac{(n + 2)}{4}$$

This gives $a_1 = \frac{5}{72}$, $a_2 = a_3 = \frac{8}{72}$.

$$\therefore \quad p_3(n) = \frac{n^2}{12} - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{8}{72}\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + \frac{8}{72}\left(\frac{-1 - i\sqrt{3}}{2}\right)^n$$

$$= \frac{n^2}{12} + \frac{1}{72}\left(16 \cos\left(\frac{2n\pi}{3}\right) - 7 - 9(-1)^n\right).$$

But $p_3(n)$ is an integer for all n, and so as

$$\max 16\left(\cos\left(\frac{2n\pi}{3}\right) - 7 - 9(-1)^n\right) = 18, \text{ for } n = 3 \text{ (say,}$$

we have $p_3(n) = \left[\frac{n^2}{12} + \frac{18}{72}\right] = \left[\frac{n^2 + 3}{12}\right]$.

<u>$m = 4$</u>

$$\therefore \quad p_4(n) = a_1 + a_2(-1)^n + a_3(i)^n + a_4(-i)^n + \frac{(n+3)^3}{144} - \frac{7n}{288}$$

$$- \frac{(n+3)^2}{24} + \frac{(n+3)}{18} + \frac{1}{8} \cdot \frac{n(-1)^n}{4} + \frac{8}{72} \cdot \frac{\left(\frac{-1 + i\sqrt{3}}{2}\right)^n}{\left(\frac{-1 + i\sqrt{3}}{2} - 1\right)}$$

$$+ \frac{8}{72} \cdot \frac{\left(\frac{-1 - i\sqrt{3}}{2}\right)^n}{\left(\frac{-1 - i\sqrt{3}}{2} - 1\right)}$$

which can be reduced to

$$p_4(n) = a_1 + a_2(-1)^n + a_3(i)^n + a_4(-i)^n + \left(\frac{2n^3 + 6n^2 - 9n + 9n(-1)^n - 6}{288}\right)$$

$$+ \frac{1}{54}\left(-6 \cos \frac{2n\pi}{3} + 2\sqrt{3} \sin \frac{2n\pi}{3}\right).$$

Whence $a_1 = -\frac{7}{288}$, $a_2 = \frac{9}{288}$, $a_3 = a_4 = \frac{1}{16}$.

$$\therefore \quad p_4(n) = -\frac{7}{288} + \frac{9}{288}(-1)^n + \frac{1}{16}(i)^n + \frac{1}{16}(-i)^n$$

$$+ \left(\frac{2n^3 + 6n^2 - 9n + 9n(-1)^n - 6}{288}\right)$$

$$+ \frac{1}{54}\left(-6 \cos \frac{2n\pi}{3} + 2\sqrt{3} \sin \frac{2n\pi}{3}\right).$$

Now following the previous technique, since $p_4(n)$ is an integer for all n, we have, for $n = 4$ (say):

$$\max\left(\frac{9}{288}(-1)^n + \frac{1}{16}(i)^n + \frac{1}{16}(-i)^n - \frac{6}{54} \cos \frac{2n\pi}{3} + \frac{2\sqrt{3}}{54} \sin \frac{2n\pi}{3}\right) = \frac{77}{288}.$$

$$\therefore \quad p_4(n) = \left[\frac{n^3 + 3n^2 + \frac{1}{2}(9n(-1)^n - 9n) + 32}{144}\right]$$

It is clear from the above form for $p_4(n)$ which contains $\cos \frac{2n\pi}{3}$ and $\sin \frac{2n\pi}{3}$ that we need to convert formula 2.4 to a form which encompasses this type in order to proceed to determine $p_m(n)$ for $m \geqslant 5$ exactly. The resulting formulas are in themselves interesting. If $\alpha^s = 1$, then

$$\alpha = \cos\left(\frac{2k\pi}{s}\right) + i \sin\left(\frac{2k\pi}{s}\right) \quad \text{and} \quad \alpha^n = \cos\left(\frac{2kn\pi}{s}\right) + i \sin\left(\frac{2kn\pi}{s}\right),$$
$$0 \leqslant k \leqslant s - 1.$$

We have from 2.4 that

$$\frac{1}{\Delta(m)}\{\alpha^n f\} = \frac{\alpha^n}{\alpha^m - 1}\left(1 - \left(\frac{\alpha^m}{\alpha^m - 1}\right)\Delta + \left(\frac{\alpha^m}{\alpha^m - 1}\right)^2 \Delta^2 - \cdots\right)\{f\}, \text{ if } \alpha^m \neq 1.$$

Then it can be shown that

$$\frac{1}{\Delta(m)}\left\{\cos\left(\frac{2kn\pi}{s}\right)f(n)\right\} = \sum_{r=0}^{p} \frac{\operatorname{cosec}^{r+1}\left(\frac{km\pi}{s}\right)}{2^{r+1}} \sin\left(\frac{k\pi}{s}(2n - m + rm) - \frac{r\pi}{2}\right)(-\Delta)^r\{f(n)\},$$

where $f(n)$ is a polynomial of degree p and $\alpha^s = 1$ but $\alpha^m \neq 1$ and $1 \leqslant k \leqslant s - 1$, $k \neq 0$. The proof is easy but lengthy.

Similarly,

$$\frac{1}{\Delta(m)}\left\{\sin\left(\frac{2kn\pi}{s}\right)f(n)\right\} = -\sum_{r=0}^{p} \frac{\operatorname{cosec}^{r+1}\left(\frac{km\pi}{s}\right)}{2^{r+1}} \cos\left(\frac{k\pi}{s}(2n - m + rm) - \frac{r\pi}{2}\right)(-\Delta)^r\{f(n)\}.$$

$\underline{m = 5}$

Thus returning to $p_5(m)$ it can be shown using the previous formulas that

$$\frac{1}{\Delta(5)}\{p_4(n+4)\} = \frac{1}{288}\left(\frac{n^4}{10} + n^3 + n^2 - 7\tfrac{1}{2}n + \frac{9(n+4)(-1)^n}{-2} + \frac{45(-1)^5}{2} \cdot \frac{(-1)^n}{(-1)^5 - 1}\right)$$
$$+ \frac{9}{288} \cdot \frac{(-1)^n}{-2} + \frac{i^n(-1 - i)}{32} + \frac{(-i)^n(-1 + i)}{32} - \frac{1}{18} \cos \frac{2n\pi}{3}$$
$$+ \frac{1}{18}\left(-\frac{1}{\sqrt{3}}\right)\sin \frac{2n\pi}{3} - \frac{\sqrt{3}}{54} \sin\left(\frac{2n\pi}{3}\right) + \frac{1}{54} \cos\left(\frac{2n\pi}{3}\right).$$

Using

$$\frac{1}{32}\left(i^n(-1 - i) + (-i)^n(-1 + i)\right) = \frac{1}{16}\left(\sin \frac{n\pi}{2} - \cos \frac{n\pi}{2}\right)$$

(continued)

$$= \frac{1}{288}\left(\frac{n^4}{10} + n^3 + n^2 - 7\tfrac{1}{2}n - \frac{9(-1)^n}{4}(2n + 5)\right) + \frac{1}{16}\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right)$$
$$- \frac{1}{27}\cos\frac{2n\pi}{3} - \frac{\sqrt{3}}{27}\sin\frac{2n\pi}{3}.$$

$\therefore$ $p_5(n)$ = C.F. + P.S., where the complementary function is

$$\sum_{k=0}^{4} a_k\left(\cos\frac{2kn\pi}{5} + i\sin\frac{2kn\pi}{5}\right),$$

which by modifying the constants a_k can clearly be written in the form

$$C_0 + C_1\cos\frac{2n\pi}{5} + C_2\cos\frac{4n\pi}{5} + S_1\sin\frac{2n\pi}{5} + S_2\sin\frac{4n\pi}{5}.$$

The method is clearly general.

$n = 0$

$$\therefore\quad C_0 + C_1 + C_2 = \frac{2395}{17{,}280} = \beta_0 \text{ (say)}$$

$n = 1$

$$\therefore\quad C_0 + C_1\cos\frac{2\pi}{5} + C_2.-\cos\frac{\pi}{5} + S_1\sin\frac{2\pi}{5} + S_2\sin\frac{\pi}{5} = -\frac{1061}{17{,}280} = \beta_1$$

$n = 2$

$$\therefore\quad C_0 + C_1.-\cos\frac{\pi}{5} + C_2\cos\frac{2\pi}{5} + S_1\sin\frac{\pi}{5} + S_2.-\sin\frac{2\pi}{5} = -\frac{1061}{17{,}280} = \beta_2$$

$n = 3$

$$\therefore\quad C_0 + C_1.-\cos\frac{\pi}{5} + C_2\cos\frac{2\pi}{5} + S_1.-\sin\frac{\pi}{5} + S_2\sin\frac{2\pi}{5} = -\frac{1061}{17{,}280} = \beta_3$$

$n = 4$

$$\therefore\quad C_0 + C_1\cos\frac{2\pi}{5} + C_2.-\cos\frac{\pi}{5} + S_1.-\sin\frac{2\pi}{5} + S_2.-\sin\frac{\pi}{5} = -\frac{1061}{17{,}280} = \beta_4$$

Thus if we add the equations we have immediately

$$C_0 = \frac{1}{5}(\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4) = \frac{-1849}{17{,}280 \times 5}.$$

As we are concerned with the mth roots of unity this form will be quite general for C_0. The solution is

$$C_1 = \frac{6912}{17{,}208 \times 5} = C_2 \text{ and } S_1 = S_2 = 0.$$

$$\therefore\quad p_5(n) = \frac{1}{2880}(n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n) + \left\{\frac{1}{288}\left(\frac{-45(-1)^n}{4}\right)\right.$$
$$+ \frac{1}{16}\left(\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right) - \frac{1}{27}\cos\frac{2n\pi}{3} - \frac{\sqrt{3}}{27}\sin\frac{2n\pi}{3} - \frac{1849}{17{,}280 \times 5}$$
$$\left. + \frac{6912}{17{,}280 \times 5}\cos\frac{2n\pi}{5} + \frac{6912}{17{,}280 \times 5}\cos\frac{4n\pi}{5}\right\}.$$

Again we have that the part within braces is purely trigonometric and has a maximum value given by $n = 5$ (say), which is 905/2880.

$$\therefore \quad p_5(n) = \left[\frac{n^4 + 10n^3 + 10n^2 - 75n - 45n(-1)^n + 905}{2880}\right]$$

It would appear from previous work that we have to determine a solution to a set of linear equations each time we determine $p_m(n)$. But this is not the case as the constants C_i and S_i can be determined explicitly in terms of the β_i as follows.

We have for the Complementary function

$$\sum_{k=0}^{m-1} a_k e^{i\frac{2kn\pi}{m}}$$

and for the Complete Solution, we have

$$n = 0 \qquad a_0 + a_1 + a_2 + \cdots + a_{m-1} = \beta_0 \text{ (say)}$$

$$n = 1 \qquad a_0 + a_1 e^{i\frac{2\pi}{m}} + a_2 e^{i\frac{4\pi}{m}} + \cdots + a_{m-1} e^{i\frac{2(m-1)\pi}{m}} = \beta_1$$

$$n = m - 1 \qquad a_0 + a_1 e^{i\frac{2(m-1)\pi}{m}} + a_2 e^{i\frac{2(m-1)2\pi}{m}} + \cdots + a_{m-1} e^{i\frac{2(m-1)(m-1)\pi}{m}} = \beta_{m-1}$$

Now $1 + e^{i\frac{2\pi r}{m}} + e^{i\frac{2r2\pi}{m}} + \cdots + e^{i\frac{2r(m-1)\pi}{m}} = \dfrac{e^{i2\pi r} - 1}{e^{i\frac{2r\pi}{m}} - 1} = 0$, as r is an integer $< m$.

Thus, if we add,

$$a_0 = \frac{\beta_0 + \beta_1 + \cdots + \beta_{m-1}}{m}.$$

To determine a_1, we can essentially do the same thing. Multiply equation (2) by $e^{-i\frac{2\pi}{m}}$, (3) by $e^{-i\frac{4\pi}{m}}$, ..., (m) by $e^{-i\frac{2(m-1)\pi}{m}}$. Thus, the coefficients in the a_1 column are all one. Then add the equations by columns again and we have

$$ma_1 = \beta_0 + \beta_1 e^{-i\frac{2\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)\pi}{m}}.$$

In general,

$$ma_k = \beta_0 + \beta_1 e^{-i\frac{2k\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)k\pi}{m}}.$$

Thus, we have the form

$$\frac{1}{m}\sum_{k=0}^{m-1}\left(\beta_0 + \beta_1 e^{-i\frac{2k\pi}{m}} + \cdots + \beta_{m-1} e^{-i\frac{2(m-1)k\pi}{m}}\right) e^{i\frac{2kn\pi}{m}}.$$

This is the Complementary function but not in an explicit real form, but the terms can be grouped to give the real form.

If m is odd $\geqslant 3$,

$$= \frac{1}{m}(\beta_0 + \beta_1 + \cdots + \beta_{m-1}) + \frac{2}{m}\sum_{k=0}^{m-1}\beta_k\left\{\cos\left(\frac{2n\pi}{m} - \frac{2k\pi}{m}\right) + \cdots + \cos\left(\frac{(m-1)n\pi}{m} - \frac{(m-1)k\pi}{m}\right)\right\}$$

If m is even $\geqslant 4$, there is a root (-1) in the form, and we have

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1}) + \frac{(-1)^n}{m}(\beta_0 - \beta_1 + \beta_2 - \cdots - \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=0}^{m-1}\beta_k\left\{\cos\left(\frac{2n\pi}{m} - \frac{2k\pi}{m}\right) + \cdots + \cos\left(\frac{(m-2)n\pi}{m} - \frac{(m-2)k\pi}{m}\right)\right\}.$$

Or finally, by regrouping, we have for m odd $\geqslant 3$:

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-1}{2}}\left(\beta_0 + \beta_1 \cos\frac{2k\pi}{m} + \cdots + \beta_{m-1}\cos\frac{2(m-1)k\pi}{m}\right)\cos\frac{2nk\pi}{m}$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-1}{2}}\left(\beta_1 \sin\frac{2k\pi}{m} + \cdots + \beta_{m-1}\sin\frac{2(m-1)k\pi}{m}\right)\sin\frac{2nk\pi}{m}.$$

For m even $\geqslant 4$,

$$= \frac{1}{m}(\beta_0 + \cdots + \beta_{m-1}) + \frac{(-1)^n}{m}(\beta_0 - \beta_1 + \cdots - \beta_{m-1})$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-2}{2}}\left(\beta_0 + \beta_1 \cos\frac{2k\pi}{m} + \cdots + \beta_{m-1}\cos\frac{2(m-1)k\pi}{m}\right)\cos\frac{2nk\pi}{m}$$
$$+ \frac{2}{m}\sum_{k=1}^{\frac{m-2}{2}}\left(\beta_1 \sin\frac{2k\pi}{m} + \cdots + \beta_{m-1}\sin\frac{2(m-1)k\pi}{m}\right)\sin\frac{2nk\pi}{m}.$$

Thus, returning to $p_6(n)$, we have that the particular solution is

$$\frac{1}{2880 \times 30}\left(n^5 + \frac{45n^4}{2} + \frac{380n^3}{3} - \frac{225n^2}{2} - 1599\tfrac{1}{6}n\right)$$
$$+ \frac{3(-1)^n}{8 \times 288}(n^2 + 9n - 39) - \frac{1}{32}\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) + \frac{n\cos\frac{2n\pi}{3}}{81}$$
$$+ \frac{6912}{17{,}280 \times 10}\left(-\operatorname{cosec}\frac{\pi}{5}\sin\frac{\pi}{5}(2n-6)\right) + \left(\operatorname{cosec}\frac{2\pi}{5}\sin\frac{2\pi}{5}(2n-6)\right).$$

The Complementary function is

$$C_0 + C_1 \cos \frac{2n\pi}{6} + C_2 \cos \frac{4n\pi}{6} + C_3 \cos \frac{6n\pi}{6} + S_1 \sin \frac{2n\pi}{6} + S_2 \sin \frac{4n\pi}{6}.$$

The coefficients C_i are

$$C_0 = -\frac{756\frac{3}{4}}{86{,}400}, \quad C_1 = C_2 = \frac{4800}{86{,}400}, \quad C_3 = \frac{5156\frac{1}{4}}{86{,}400}$$

$$S_1 = 0, \qquad S_2 = \frac{1066\frac{2}{3}}{\sqrt{3} \times 86{,}400}$$

Thus $p_6(n)$ is the sum of the two forms. Again the maximum value of the purely trigonometric part—that is, the part that does not contain any algebraic powers of n, is given when $n = 6$ and is 19,224/86,400. Hence,

$$p_6(n) = \left[\frac{n^5 + 22\frac{1}{2}n^4 + 126\frac{2}{3}n^3 - 112\frac{1}{2}n^2 - 1599\frac{1}{6}n + 112\frac{1}{2}(-1)^n(n^2+9n) + 1066\frac{2}{3}n\cos\frac{2n\pi}{3} + 19224}{6!5!}\right].$$

The method can of course be continued; I simply state the result for $p_7(n)$.

$$p_7(n) = \left[\frac{\left(n^6 + 42n^5 + 560n^4 + 1960n^3 - 8725\frac{1}{2}n^2 - 45{,}325n - (-1)^n \cdot 2362\frac{1}{2}(n^2+14n) + 22{,}400n \operatorname{cosec}\frac{\pi}{3}\sin\frac{\pi}{3}(2n-7) + 1{,}029{,}154\right)}{7!6!}\right].$$

Having determined the explicit form for $p_6(n)$, it is time for some general remarks. Looking at the method of production, we can see that the leading terms are purely algebraic and that this property of the formulas will continue under the operator $\frac{1}{\Delta(m)}$. The leading nonalgebraic power of n or, more precisely, its coefficient increases when (-1) is a root of the operator $\frac{1}{\Delta(m)}$, as we see from formula 2.5.

That is for all even powers of m. Thus for $m = 7$ we have that the first four powers are purely algebraic, that is, for n^6, n^5, n^4, and n^3. For $n = 8$ we have that n^7, n^6, n^5, and n^4 will be, but not n^3.

The pattern is quite clear, and we can see that the first $\left[\frac{m+1}{2}\right]$ powers are purely algebraic in $p_m(n)$. We can go further than this and say that $p_m(n)$ contains a purely algebraic part which is a polynomial in n of degree $(m - 1)$ with rational coefficients as the Bernoulli numbers B_i are rational. Let this polynomial of degree $(m - 1)$ be denoted by $q_m(n)$ (say) and the trigonometric or nonpolynomial part by $t_m(n)$. Thus

$$p_m(n) = q_m(n) + t_m(n),$$

where the polynomials $q_m(n)$ naturally satisfy

$$q_m(n) - q_m(n - m) = q_{m-1}(n - 1).$$

From the forms so far determined, we have

$$q_2(n) = \frac{1}{2!1!}(n - \tfrac{1}{2})$$

$$q_3(n) = \frac{1}{3!2!}(n^2 - 1\tfrac{1}{6})$$

$$q_4(n) = \frac{1}{4!3!}(n^3 + 3n^2 - 4\tfrac{1}{2}n - 6\tfrac{1}{2})$$

$$q_5(n) = \frac{1}{5!4!}(n^4 + 10n^3 + 10n^2 - 75n - 61\tfrac{19}{30})$$

$$q_6(n) = \frac{1}{6!5!}(n^5 + 22\tfrac{1}{2}n^4 + 126\tfrac{2}{3}n^3 - 112\tfrac{1}{2}n^2 - 1599\tfrac{1}{6}n - 756\tfrac{3}{4})$$

where the constant term is just the value of C_0. As the first $\left[\frac{m+1}{2}\right]$ terms agree with $p_m(n)$, an examination of the general form of these leading terms is required.

3. A SERIES EXPANSION FOR $q_m(n)$

The general form for the leading terms of $q_m(n)$ are given in [1], where I also consider the problem of determining an upper bound for $p_m(n)$ for arbitrary m and n, together with some numerical examples. For the sake of completeness, I simply quote the expansion of $q_m(n)$ given in that paper.

$$\begin{aligned} q_m(n) = {} & \frac{n^{m-1}}{m!(m-1)!} + \frac{1}{m!(m-2)!}\left(\frac{m^2 - 3m}{4 \cdot 1!}\right)n^{m-2} \\ & + \frac{1}{m!(m-3)!}\left(\frac{m^4 - \frac{58}{9}m^3 + \frac{75}{9}m^2 - \frac{2}{9}m}{4^2 \cdot 2!}\right)n^{m-3} \\ & + \frac{1}{m!(m-4)!}\left(\frac{m^6 - \frac{31}{3}m^5 + 29m^4 - \frac{65}{3}m^3 + 2m^2}{4^3 \cdot 3!}\right)n^{m-4} \\ & + \frac{1}{m!(m-5)!}\left[\frac{\left(m^8 - 14\frac{2}{3}m^7 + 66\frac{16}{27}m^6 - 107\frac{29}{225}m^5 + 55\frac{134}{135}m^4 - 10\frac{54}{135}m^3 + \frac{4}{27}m^2 - \frac{16}{225}m\right)}{4^4 \cdot 4!}\right]n^{m-5} \end{aligned}$$

where the first $\left[\frac{m+1}{2}\right]$ terms in the expansion of $p_m(n)$ are algebraic and agree with the terms above if $\left[\frac{m+1}{2}\right] \geqslant 5$ or $m \geqslant 9$. The polynomials can be generated by means of a computer program where the summations are effected using the Bernoulli polynomials. This expansion, although of some interest, is of little use for calculating $p_m(n)$ unless n is large compared with m. J. W. L. Glaisher gives an expansion for $q_m(n)$ based on the "waves" of J. J. Sylvester (see Gupta [3]).

Looking at the action of the operator $1/\Delta(m)$ in formulas 2.2 and 2.3, it is easy to see the form of the leading term in $t_m(n)$, the nonpolynomial part of $p_m(n)$. We have

$$t_m(n) = \frac{(-1)^{m+n} n^{\left[\frac{m-2}{2}\right]}}{2^m \left[\frac{m}{2}\right]! \left[\frac{m-2}{2}\right]!} \ \ldots \text{ for } m \geqslant 4.$$

4. CONCLUSION

The method not only yields closed formulas for small values of but also illustrates the general structure of $p_m(n)$. The method is perfectly general but clearly, as the formulas are calculated recurvisely, the computations become increasingly lengthy. The method can also be used to determine closed formulas for partitioning into an arbitrary small set of integers. The recurrence relationship is

$$p^\star(p_1, p_2, \ldots, p_m; n) - p^\star(p_1, p_2, \ldots, p_m; n - p_m) = p^\star(p_1, p_2, \ldots, p_{m-1}; n)$$

where $p^\star(p_1, p_2, \ldots, p_m; n)$ means the number of partitions of n into at most parts $p_1, p_2, \ldots, p_m$ or, equivalently, the number of solutions in integers $\geqslant 0$ of the Diophantine equation

$$p_1x_1 + p_2x_2 + \cdots + p_mx_m = n.$$

For example, the method yields

$$p^\star(1, 2, 3, 5; n) = \left[\frac{n^3 + 16\frac{1}{2}n^2 + 81n + 180}{1 \cdot 2 \cdot 3 \cdot 5 \cdot 3!}\right].$$

This more general problem will be explored in a future paper.

REFERENCES

1. W. J. A. Colman. "The Number of Partitions of the Integer n into m Non-Zero Positive Integers." *Math. of Comp.*, July 1982.
2. L. H. Dickson. *History of the Theory of Numbers*. II. Chelsea, N.Y., 1966.
3. H. Gupta. *The Royal Society Mathematical Tables*. IV. Cambridge, 1958.

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