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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. HILLMAN

Assistant Editors: Gloria C. Padilla & Charles R. Wall

Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date, and proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-508 *Proposed by Philip L. Mana, Albuquerque, NM*

Find all n in $\{1, 2, 3, \ldots, 200\}$ such that the sum $n! + (n + 1)!$ of successive factorials is the square of an integer.

B-509 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let ψ be Dedekind's function given by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

For example, $\psi(12) = 12\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) = 24$. Show that

$$\psi(\psi(\psi(n))) > 2n \text{ for } n = 1, 2, 3, \ldots .$$

B-510 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Euler's ϕ function and its companion, Dedekind's ψ function are defined by

$$\phi(n) = n \prod_{p|n}\left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(n) = n \prod_{p|n}\left(1 + \frac{1}{p}\right).$$

(a) Show that $\phi(n) + \psi(n) \geqslant 2n$ for $n > 1$.
(b) When is the inequality strict?

B-511 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j even. Prove that

$$F_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = (L_{n+2jk+j} - L_{n-j})/5.$$

B-512 *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, and n be integers with j odd. Prove that

$$L_j(F_n + F_{n+2j} + F_{n+4j} + \cdots + F_{n+2jk}) = F_{n+2jk+j} - F_{n-j}.$$

B-513 *Proposed by Andreas N. Philippou, University of Patras, Greece*

Show that

$$\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = \sum_{k=0}^{[n/2]} (n+1-k)\binom{n-k}{k} \text{ for } n = 0, 1, \ldots,$$

where $[x]$ denotes the greatest integer in x.

SOLUTIONS

Correction of a Previously Published "Solution"

B-468 *Proposed by Miha'ly Bencze, Brasov, Romania*

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

Solution by Charles R. Wall, Trident Technical College, Charleston, SC

The published solution (*FQ*, Feb. 1983) is clearly erroneous, because it allows negative terms in a sequence of positive numbers. The error apparently arises from $(1 - \sin^2 t)^{\frac{1}{2}} = \cos t$, which is false if $\cos t < 0$.

Let

$$b_n = F_{n-2} \text{ Arcsin } a_1 + F_{n-1} \text{ Arcsin } a_2$$

and let k be the least positive integer for which $b_k > \pi/2$. Then $k \geqslant 3$, and it is easy to show that $a_n = \sin b_n$ for $n \leqslant k$ (as given in the erroneous solution). However,

$$\begin{aligned} a_{k+1} &= \sin b_k(1 - \sin^2 b_{k-1})^{\frac{1}{2}} + \sin b_{k-1}(1 - \sin^2 b_k)^{\frac{1}{2}} \\ &= \sin b_k(\cos b_{k-1}) + \sin b_{k-1}(-\cos b_k) \\ &= \sin(b_k - b_{k-1}) = \sin b_{k-2} = a_{k-2}. \end{aligned}$$

Also,

$$\begin{aligned} a_{k+2} &= \sin b_{k-2}(1 - \sin^2 b_k)^{\frac{1}{2}} + \sin b_k(1 - \sin^2 b_{k-2})^{\frac{1}{2}} \\ &= \sin b_{k-2}(-\cos b_k) + \sin b_k(\cos b_{k-2}) \\ &= \sin(b_k - b_{k-2}) = \sin b_{k-1} = a_{k-1}. \end{aligned}$$

Then

$$\begin{aligned} a_{k+3} &= a_{k+2}(1 - a_{k+1}^2)^{\frac{1}{2}} + a_{k+1}(1 - a_{k+2}^2)^{\frac{1}{2}} \\ &= a_{k-1}(1 - a_{k-2}^2)^{\frac{1}{2}} + a_{k-2}(1 - a_{k-1}^2)^{\frac{1}{2}} = a_k. \end{aligned}$$

Thus, the sequence eventually repeats in a cycle of three values, so we have

$$a_n = \begin{cases} \sin b_n & \text{if } n \leqslant k \\ \sin b_{k-2} & \text{if } n = k + 3j + 1 \text{ and } j \geqslant 0 \\ \sin b_{k-1} & \text{if } n = k + 3j + 2 \text{ and } j \geqslant 0 \\ \sin b_k & \text{if } n = k + 3j \text{ and } j \geqslant 0 \end{cases}$$

where $\{b_n\}$ and k are defined as above.

Efficient Raising to Powers

B-484 *Proposed by Philip L. Mana, Albuquerque, NM*

For a given x, what is the least number of multiplications needed to calculate x^{98}? (Assume that storage is unlimited for intermediate products.)

Solution by Walther Janous, Universitaet Innsbruck, Austria

Since $96 = 2^6 + 2^5 + 2$, the least number of multiplications needed to calculate x^{98} is $6 + 2 = 8$. This can be achieved as follows:

$$xx = x^2;\ x^2x^2 = x^4;\ x^4x^4 = x^8;\ x^8x^8 = x^{16};\ x^{16}x^{16} = x^{32};$$

$$x^{32}x^{32} = x^{64};\ x^{32}x^{64} = x^{96};\ x^{96}x^2 = x^{98}.$$

In general, the following theorem holds true: If

$$\sum_{i=1}^{k} a_i 2^i,\ a_i \in \{0, 1\},$$

is the dual-representation of the number N, then the least number of multiplications needed to calculate x^N (under assumption of unlimited storage for intermediate products) equals

$$p(N) = k + \#\{i : i < k \text{ and } a_i = 1\}.$$

Also solved by L. Kuipers, Vania D. Mascioni, Samuel D. Moore, John Oman & Bob Prielipp, Stanley Rabinowitz, Sahib Singh, J. Suck, and the proposer.

Difference Equation

B-485 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Find the complete solution u_n to the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = 11F_n - 4F_{n+2}.$$

Solution by J. Suck, Essen, Germany

Since

$$u_{n+2} - 5u_{n+1} + 6u_n = 11F_n - 4F_{n+2} = F_{n+2} - 5F_{n+1} + 6F_n,$$

we see that the difference sequence $d_n := u_n - F_n$ has the auxiliary equation $x^2 - 5x + 6 = 0$, of which the roots are 2 and 3. The general solution for d_n is, thus, $d_n = a2^n + b3^n$, and so $u_n = a2^n + b3^n + F_n$ with arbitrary constants a, b [which are $a = 3(u_0 - F_0) - u_1 + F_1$, $b = u_1 - F_1 - 2(u_0 - F_0)$ in terms of initial values].

Of course, the solution does not depend on $F_0 = 0$, $F_1 = 1$, but only on the Fibonacci recurrence.

Also solved by Wray G. Brady, Paul S. Bruckman, C. Georghiou, Walther Janous, L. Kuipers, John W. Milsom, Bob Prielipp, A. G. Shannon, Sahib Singh, and the proposer.

Monotonic Sequences of Ratios

B486 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Prove or disprove that, for every positive integer k,

$$\frac{F_{k+1}}{F_1} < \frac{F_{k+3}}{F_3} < \frac{F_{k+5}}{F_5} < \cdots < \alpha^k < \cdots < \frac{F_{k+6}}{F_6} < \frac{F_{k+4}}{F_4} < \frac{F_{k+2}}{F_2}.$$

Solution by Vania D. Mascioni, student, Swiss Fed. Inst. of Tech., Zürich

Fix $k > 0$. Using the well-known identity

$$F_{n+k}F_{m-k} - F_nF_m = (-1)^n F_{m-n-k}F_k$$

(see, e.g., Knuth, *The Art of Computer Programming*, I, Ex. 1.2.8.17), we obtain

$$F_{k+2P}F_{2P+2} - F_{k+2P+2}F_{2P} - F_{k+2P+1}F_{2P-1} - F_{k+2P-1}F_{2P+1} = F_k > 0.$$

It is then $\dfrac{F_{k+2P+2}}{F_{2P+2}} < \dfrac{F_{k+2P}}{F_{2P}}$ and $\dfrac{F_{k+2P+1}}{F_{2P+1}} > \dfrac{F_{k+2P-1}}{F_{2P-1}}$ for $P \geqslant 1$.

From $F_n = \left[\frac{\alpha^n}{5} + \frac{1}{2}\right]$, it follows that

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \alpha^k.$$

Also solved by Paul S. Bruckman, C. Georghiou, Walther Janous, L. Kuipers, Bob Prielipp, Stanley Rabinowitz, A. G. Shannon, Sahib Singh, J. Suck, and the proposer.

Multiple of 50

B-487 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that, for all positive integers n,

$$5L_{4n} - L_{2n}^2 + 6 - 6(-1)^n L_{2n} \equiv 0 \pmod{10F_n^2}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We will show that the given congruence holds. Since

$$L_{2n} = 5F_n^2 + 2(-1)^n,\ F_{2n} = L_nF_n,\ \text{and}\ L_n^2 - F_n^2 = 4F_n^2 + 4(-1)^n$$

(See Exercises 4, 1, and 10 on p. 29 of *Fibonacci and Lucas Numbers* by V. E. Hoggatt, Jr.),

$$5L_{4n} - L_{2n}^2 + 6 - 6(-1)^n L_{2n} = 25F_{2n}^2 + 10 - [5F_n^2 + 2(-1)^n]^2 + 6 - 6(-1)^n$$

$$[5F_n^2 + 2(-1)^n] = 25F_{2n}^2 + 10 - 25F_n^4 - 20(-1)^n F_n^2 - 4 + 6 - 30(-1)^n F_n^2 - 12$$

$$= 25L_n^2F_n^2 - 25F_n^4 - 50(-1)^n F_n^2 = 25F_n^2(L_n^2 - F_n^2) - 50(-1)^n F_n^2$$

$$= 25F_n^2[4F_n^2 + 4(-1)^n] - 50(-1)^n F_n^2 = 50F_n^2[2F_n^2 + (-1)^n].$$

Clearly the immediately preceding expression is congruent to zero modulo $50F_n^2$ (and hence is congruent to zero modulo $10F_n^2$).

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Stanley Rabinowitz, Heinz-Jürgen Seiffert, Sahib Singh, J. Suck, and the proposer.

Odd Difference

B-488 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a and d be positive integers with d odd. Prove or disprove that for all positive integers h and k,

$$L_{a+hd} + L_{a+hd+d} \equiv L_{a+kd} + L_{a+kd+d} \pmod{L_d}.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

This congruence is true. The proof follows by using the result of B-479 which states that

$$L_{a+hd} + L_{a+hd+d} \equiv L_{a+d} + L_a \pmod{L_d}.$$

Similarly,

$$L_{a+kd} + L_{a+kd+d} \equiv L_{a+d} + L_a \pmod{L_d}$$

is true.

By subtraction, the required result follows, and we are done.

Also solved by Paul S. Bruckman, Walther Janous, L. Kuipers, Bob Prielipp, J. Suck, and the proposer.

Even Difference

B-489 *Proposed by Herta T. Freitag, Roanoke, VA*

Is there a Fibonacci analogue (or semianalogue) of B-488?

Solution by Walther Janous, Universitaet Innsbruck, Austria

Let a and d be positive integers with d *even*. Then there holds for all positive integers h and k,

$$F_{a+hd} + F_{a+hd+d} \equiv F_{a+kd} + F_{a+kd+d} \pmod{F_d}.$$

As before, it is enough to consider the case $h = k + 1$. Since, for d even, there holds

$$F_{a+(k+2)d} - F_{a+kd} = F_d L_{a+(k+1)d},$$

the claim is proved.

Also solved by Paul S. Bruckman, L. Kuipers, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

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