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THE FIBONACCI SEQUENCE F_n MODULO L_m

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This paper is concerned with determining the length of the period of a Fibonacci series after reducing it by a modulus m. Some of the results established by Wall (see [1]) are used. We investigate further the length of the period.

The Fibonacci sequence is defined with the conditions $f_0 = \alpha$, $f_1 = \beta$ and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$. We will refer to the two special sequences when $\alpha = 0$, $\beta = 1$ and $\alpha = 2$, $\beta = 1$ as (F_n) and (L_n), respectively. (L_n) is often called the Lucas sequence.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... reduced modulo 3 is

$$0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1,\ 1,\ 2,\ \ldots .$$

The reduced sequence repeats after 8 terms. We say that the reduced sequence is periodic with period 8. The second half of the period is twice the first half. We refer to the terminology used by Robinson [2] and say that the sequence has a restricted period of 4 with multiplier 2 or -1 (since $2 \equiv -1$ mod 3). If the reduced sequence has a value of -1 at F_{k-1} and 0 at F_k, then the sequence is said to have a restricted period of k with multiplier -1. The period of the reduced sequence is $2k$. The $2k$ terms of the period form two sets of k terms. The terms of the second half are -1 times the terms of the first half.

Wall [1] produced many results concerning the length of the period of the recurring sequence obtained by reducing a Fibonacci sequence by a modulus m. The length of the period of the special sequence F_n reduced modulo m will be denoted by $p(m)$.

Theorem 1 (Wall)

f_n (mod m) forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.

We have (see [3]):

(1) $F_m = (a^m - b^m)/(a - b)$,

(2) $L_m = a^m + b^m = F_{m-1} + F_{m+1}$, where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$.

Also,

(3) $F_{2m} \equiv 0 \pmod{L_m}$ [follows from (1) and (2)].

Note that

$$ab = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 - \sqrt{5}}{2}\right) = -1.$$

Since $(ab)^{m-1} = (-1)^{m-1}$, we have

$$\begin{aligned} a^{2m-1} - b^{2m-1} - (-1)^{m-1}(a - b) &= a^{2m-1} - b^{2m-1} - (ab)^{m-1}(a - b) \\ &= a^{2m-1} - b^{2m-1} - a\, b^{m-1} + a^{m-1}b^m \\ &= (a^{m-1} - b^{m-1})(a^m + b^m). \end{aligned}$$

From this, we have

$$F_{2m-1} - (-1)^{m-1} = F_{m-1}L_m.$$

Hence

(4) $F_{2m-1} \equiv (-1)^{m-1} \pmod{L_m}$.

Theorem 2

For $m \geqslant 2$, the Fibonacci sequence $F_n \pmod{L_m}$ has period $4m$ if m is even and period $2m$ if m is odd.

Proof: Suppose m is odd, and the sequence $F_n \pmod{L_m}$ has period p. It follows from (3) and (4) that the reduced sequence has values 1 at F_{2m-1} and 0 at F_{2m}. Therefore, $2m$ is a multiple of p and $2m = kp$ for some integer $k > 0$. From (2) we have $L_m = F_{m-1} + F_{m+1}$ and $L_m > F_j$ for all $j \leqslant m + 1$, if $m \geqslant 2$. Hence, L_m cannot divide any F_j for $j \leqslant m + 1$, which implies that $F_j \not\equiv 0 \pmod{L_m}$ for any $j \leqslant m$. Therefore, $p > m$, $kp = 2m < 2p$, and $k < 2$. Thus, $k = 1$ and $p(L_m) = 2m$.

Suppose m is even. It follows from (3) and (4) that the reduced sequence has values -1 at F_{2m-1} and 0 at F_{2m}. This implies that the reduced sequence has a restricted period. Let p' be the restricted period. It follows that $2m = k \cdot p'$ for some $k > 0$. Again $m < p'$ since $F_j < L$ for all $j \leqslant m$. This implies that $k < 2$ and, therefore, $k = 1$. Thus, the restricted period is $2m$ and the period is $4m$. ■

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