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POWERS OF T AND SODDY CIRCLES

JOHN H. SELLECK
Winston-Salem, N.C. 27106

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1. INTRODUCTION

T is the real root of the equation $T^3 - T^2 - T - 1 = 0$, and is approximately equal to 1.8392867... T has the property:

$$T^{n-3} + T^{n-2} + T^{n-1} = T^n,$$

which is similar to the formula that defines the Tribonacci numbers:

$$t(n - 3) + t(n - 2) + t(n - 1) = t(n).$$

In fact, T has a relationship to the Tribonacci numbers similar to that between ϕ and the Fibonacci numbers. Binet's formula for calculating the value of the nth Fibonacci number is

$$f(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

Since $\phi^{-1} = .618... < 1$, we can see that the ratio between two adjacent Fibonacci numbers is a close approximation to ϕ, and moreso as the value of n increases:

$$f(n + 1)/f(n) = (\phi^{n+1} - (-\phi)^{-(n+1)})/(\phi^n - (-\phi)^{-n}) \to \phi \text{ as } n \to \infty.$$

Similarly, given Binet's formula for deriving a Tribonacci number $t(n)$:

$$t(n) = \alpha T^n + r^n(\beta \cos n\theta + \gamma \sin n\theta) \quad \text{(see [1])},$$

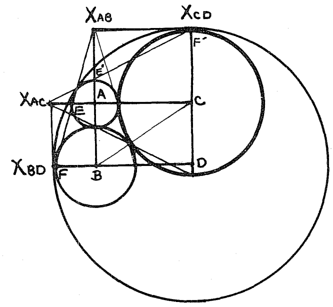
and since $|r| = .7374... < 1$, we can see that the value of the ratio of two adjacent Tribonacci numbers is a close approximation to T, and moreso as the value of n increases:

$$t(n + 1)/t(n) = (\alpha T^{n+1} + r^{n+1}(\beta \cos n\theta + \gamma \sin n\theta))/$$
$$(\alpha T^n + r^n(\beta \cos n\theta + \gamma \sin n\theta)) \to T \text{ as } n \to \infty.$$

2. A GEOMETRIC APPLICATION OF T

If three circles are externally tangent to each other, and the radii of each are three successive powers of T, then a fourth circle, internally tangent to all three has a radius equal to the next higher power of T.

Proof: Given the three circles with centers A, B, and C:



$$rA = T^n$$
$$rB = T^{n+1}$$
$$rC = T^{n+2}$$

Since $\quad (AB)^2 = (T^n + T^{n+1})^2 = T^{2n} + 2T^{2n+1} + T^{2n+2}$

and $\quad (AC)^2 = (T^n + T^{n+2})^2 = T^{2n} + 2T^{2n+2} + T^{2n+4}$,

then $\quad (AB)^2 + (AC)^2 = 2(T^{2n} + T^{2n+1} + T^{2n+2}) + T^{2n+2} + T^{2n+4}$

$$= T^{2n+2} + 2T^{2n+3} + T^{2n+4}.$$

And since $\quad (BC)^2 = (T^{n+1} + T^{n+2})^2 = T^{2n+2} + 2T^{2n+3} + T^{2n+4}$,

then $\quad (AB)^2 + (AC)^2 = (BC)^2$.

Triangle ABC is a right triangle; angle BAC = 90 degrees. Extend CA to E on the circumference of circle A. Draw BF parallel to AC; F is on the circumference of circle B. Extend FE to meet AB extended at X_{AB}, which is the external center of similitude for circles A and B.

Then, if $X_{AB}A = X$, an unknown, and

$$AE/FB = X/(X + AB)$$

and given the aforementioned values for AB, $AE = rA$, and $FB = rB$, then

$$T^n/T^{n+1} = X/(X + T^n + T^{n+1})$$

$$XT^{n+1} = XT^n + T^{2n} + T^{2n+1}$$

$$X(T^{n+1} - T^n) = T^{2n} + T^{2n+1}.$$

If we define

$$d = T^n/(T^{n+1} - T^n) = T^n/(T^{n-1} + T^{n-2}),$$

then

$$T^{n+1} - T^n = T^{n-1} + T^{n-2} = T^n/d$$

and

$$T^{2n} + T^{2n+1} = T^{2n+2}/d;$$

therefore,

$$X = (T^{2n} + T^{2n+1})/(T^{n+1} - T^n) = (T^{2n+2}/d)(T^n/d) = T^{n+2} = rC.$$

Where a tangent from X_{AB} touches the circumference of circle C is the external center of similitude between circle C and the fourth circle (X_{CD}), which is where they are internally tangent; a line drawn from X_{CD} through C will contain the center of the fourth circle, D. Since $X_{AB}A$ is perpendicular to AC and equal to rC, $X_{CD}C$ is parallel to AB and also perpendicular to AC.

We can also construct the point X_{BD} in the same manner; $X_{BD}B$ will be found to be perpendicular to AB and parallel to AC. So D is located at a point such that BD is parallel and equal to AC and perpendicular to AB and CD; AB and CD are in turn parallel and equal to each other.

The definition of the construction of this fourth circle, D, is that it is tangent to each of the other three circles at a point where a line from the external center of similitude of the other two circles in each case is tangent to it. We do not need to construct point X_{AD} to locate point D.

Therefore, since

$$rD = rC + CD = rB + BD,$$

and having shown that

$$CD = AB = rA + rB$$

and that

$$BD = AC = rA + rC,$$

then

$$rD = rA + rB + rC = T^n + T^{n+1} + T^{n+2} = T^{n+3}.$$

Q.E.D.

REFERENCE

1. W. R. Spickerman. "Binet's Formula for the Tribonacci Sequence." *The Fibonacci Quarterly* 19, No. 2 (1982):118-20.

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