◆◇◆◇◆

COUNTING THE PROFILES IN DOMINO TILING

T. C. WU
York College, CUNY, Jamaica, NY 11451

(Submitted June 1982)

1. INTRODUCTION

Read [2] describes "profiles" that can be formed when one tiles a given rectangle with dominoes. For rectangles of width $m = 2, 3, 4$, the number of profiles $N(m)$ subject to certain rules are shown to be 2, 9, and 12, respectively. In fact, it is not difficult for one to program a computer to produce the following tabulated values for $N(m)$:

m	2	3	4	5	6	7	8	9	10
$N(m)$	2	9	12	50	60	245	280	1134	1260

We notice that values of $N(m)$ grow rather rapidly. Knowing these numbers is helpful in the estimation of execution time and storage requirement if one follows Read's method to calculate the number of domino tilings on a given chessboard.

In this note, we shall sketch a proof of the following formula:

$$N(m) = \begin{cases} \dbinom{m}{m/2} m/2, & \text{if } m \text{ is even} \\[2ex] \dbinom{m+1}{(m+1)/2} m/2, & \text{if } m \text{ is odd.} \end{cases}$$

2. DEFINING THE PROFILES

The profiles in [2] can be seen as patterns on an $m \times 2$ board with certain properties. We label 1 for each square taken by a domino and label 0 for each square not taken by a domino on the profile. For $m = 4$, say, we can represent the 12 profiles in [2] as follows,

00	00	00	11	10	11	11	10	10	11	10	11
00	10	00	00	00	00	11	10	11	11	10	10
00	10	10	00	00	10	00	00	00	11	11	10
00	00	10	00	10	10	00	00	10	00	00	00
A	L	I	B	H	K	D	C	J	G	F	E
	(1)			(2)			(3)			(4)	

where the letters A-L are names of the corresponding profiles given in [2].

Count rows from top to bottom and columns from left to right. Assign Boolean variables $L_1, L_2, \ldots, L_m$ to the corresponding left squares and Boolean variables $R_1, R_2, \ldots, R_m$ to the corresponding right squares. Using the argument of [1], a profile can be defined as a solution of the following system of equations and inequalities,

$$\sum_{i=1}^{m} (-1)^{i+1}(L_i - R_i) = p$$

$$L_i \geqslant R_{i+j}, \ i = 1, \ldots, m; \ j = 0, 1, \ldots, m - i \qquad (*)$$

$$L_1 + L_2 + \cdots + L_m < m,$$

where $p = 0$ if m is even and $p = 0$ or 1 if m is odd.

3. COUNTING THE PROFILES

We shall indicate how to calculate the number of solutions of the system (*) when $m = 2h$ is even. Consider the cases,

$$C_k : L_k = 0, \text{ and } L_j = 1 \text{ for } j < k$$

for $k = 1, \ldots, m$. Then by the first inequality in (*), $R_{k+j} = 0$ for $j = 0, 1, \ldots, m - k$. For example, when $m = 4$, the four cases are shown in the previous section.

Assume the case C_k. The equation in the system (*) becomes

$$\sum_{i=1}^{k-1}(-1)^{i+1}(1 - R_i) + \sum_{i=k+1}^{m}(-1)^{i+1}L_i = 0.$$

When k is odd, there are

$$\sum_{i=0}^{h-i}\binom{h-1}{i}\binom{h}{i} \qquad (1)$$

solutions.

When k is even, there are

$$\sum_{i=0}^{h-i}\binom{h-1}{i}\binom{h}{i+1} \qquad (2)$$

solutions.

In either case, the number is independent of k. There are h odd k values and h even k values. The number of solutions of (*) is h times the sum of (1) and (2), which is the number of profiles when m is even.

4. OTHER CONNECTIONS

Klarner and Pollack [1] attacked the domino tiling problem using a different approach. It is interesting to note that the number of profiles is always $m/2$ times the dimension of the graph matrix constructed in [1]. The graph matrix obtained from the profiles has a simpler structure than the one used in [1]. The number of edges of the graph matrix in Read [2] can be calculated by the following formula:

$$E(m) = \begin{cases} N(m) \times 3/2, & \text{if } m \text{ is even} \\ N(m) \times (3/2 - 1/(2m \times m)), & \text{if } m \text{ is odd.} \end{cases}$$

We see that $E(m)$ is close to 3/2 of $N(m)$ when m is large.

ACKNOWLEDGMENT

The author would like to thank Professors B. Coen and J. Malkevitch for their interest and stimulating conversations on the domino problem.

REFERENCES

1. D. A. Klarner & J. Pollack. "Domino Tilings of Rectangles with Fixed Width." *Discrete Mathematics* 32 (1980):53-57.
2. R. C. Read. "A Note on Tiling Rectangles with Dominoes." *The Fibonacci Quarterly* 18, No. 1 (1980):24-27.

◆◇◆◇◆

THE FIBONACCI SEQUENCE F_n MODULO L_m

KAROLYN A. MORGAN
The University of Alabama in Birmingham, AL 35294

(Submitted August 1981)

This paper is concerned with determining the length of the period of a Fibonacci series after reducing it by a modulus m. Some of the results established by Wall (see [1]) are used. We investigate further the length of the period.

The Fibonacci sequence is defined with the conditions $f_0 = \alpha$, $f_1 = \beta$ and $f_{n+1} = f_n + f_{n-1}$ for $n > 1$. We will refer to the two special sequences when $\alpha = 0$, $\beta = 1$ and $\alpha = 2$, $\beta = 1$ as (F_n) and (L_n), respectively. (L_n) is often called the Lucas sequence.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... reduced modulo 3 is

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \ldots .$$

The reduced sequence repeats after 8 terms. We say that the reduced sequence is periodic with period 8. The second half of the period is twice the first half. We refer to the terminology used by Robinson [2] and say that the sequence has a restricted period of 4 with multiplier 2 or -1 (since $2 \equiv -1$ mod 3). If the reduced sequence has a value of -1 at F_{k-1} and 0 at F_k, then the sequence is said to have a restricted period of k with multiplier -1. The period of the reduced sequence is $2k$. The $2k$ terms of the period form two sets of k terms. The terms of the second half are -1 times the terms of the first half.

Wall [1] produced many results concerning the length of the period of the recurring sequence obtained by reducing a Fibonacci sequence by a modulus m. The length of the period of the special sequence F_n reduced modulo m will be denoted by $p(m)$.

Theorem 1 (Wall)

f_n (mod m) forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.