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ADVANCED PROBLEMS AND SOLUTIONS

Edited by

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-365 *Proposed by Larry Taylor, Rego Park, NY*

Call a Fibonacci-Lucas identity divisible by 5 if every term of the identity is divisible by 5. Prove that, for every Fibonacci-Lucas identity not divisible by 5, there exists another Fibonacci-Lucas identity not divisible by 5 that can be derived from the original identity in the following way:

1) If necessary, restate the original identity in such a way that a derivation is possible.

2) Change one factor in every term of the original identity from F_n to L_n or from L_n to $5F_n$ in such a way that the result is also an identity. If the resulting identity is not divisible by 5, it is the derived identity.

3) If the resulting identity is divisible by 5, change one factor in every term of the original identity from L_n to F_n or from $5F_n$ to L_n in such a way that the result is also an identity. This is equivalent to dividing every term of the first resulting identity by 5. Then, the second resulting identity is the derived identity.

For example, $F_nL_n = F_{2n}$ can be restated as

$$F_nL_n = F_{2n} \pm F_0(-1)^n.$$

This is actually two distinct identities, of which the derived identities are

$$L_n^2 = L_{2n} + L_0(-1)^n$$

and

$$5F_n^2 = L_{2n} - L_0(-1)^n.$$

H-366 *Proposed by Stanley Rabinowitz, Merrimack, NH*

The *Fibonacci polynomials* are defined by the recursion

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with the initial conditions $f_1(x) = 1$ and $f_2(x) = x$. Prove that the discriminant of $f_n(x)$ is $(-1)^{(n-1)(n-2)/2}2^{n-1}n^{n-3}$ for $n > 1$.

Remark: The idea of investigating discriminants of interesting polynomials was suggested by [1]. The definition of the discriminant of a polynomial can be found in [2]. Fibonacci polynomials are well known, see, for example, [3] and [4]. I ran a computer program to find the discriminant of $f_n(x)$ as n varies from 2 to 11, and by analyzing the results, reached the conjecture given in Problem H-366. The discriminant was calculated by finding the resultant of $f_n(x)$ and $f_n'(x)$ using a computer algebra system similar to the MACSYMA program described in [5]. Much useful material can be found in [6] where the problem of finding the discriminant of the Hermite, Laguerre, and Chebyshev polynomials is discussed. The discriminant of the Fibonacci polynomials should be provable using similar techniques; however, I was not able to do so.

References

1. Phyllis Lefton. "A Trinomial Discriminant Formula." *The Fibonacci Quarterly* 20 (1982):363-65.
2. Van der Warden. *Modern Algebra*. New York: Ungar, 1940, I, 82.
3. M. N. S. Swamy. Problem B-84. *The Fibonacci Quarterly* 4 (1966):90.
4. S. Rabinowitz. Problem H-129. *The Fibonacci Quarterly* 6 (1968):51.
5. W. A. Martin & R. J. Fateman. "The MACSYMA System." *Proceedings of the 2nd Symposium on Symbolic and Algebraic Manipulation*, Association for Computing Machinery, 1971, pp. 59-75.
6. D. K. Faddeev & I. S. Sominskii. *Problems in Higher Algebra*. Trans. J. L. Brenner. San Francisco: W. H. Freeman, 1965, Problems 833-851.

H-367 *Proposed by M. Wachtel, Zurich, Switzerland*

A. Prove the identity:

$$\sqrt{(L_{2n} - L_{n-2}^2) \cdot (L_{2n+4} - L_n^2) + 30} = 5F_{2n} - 3(-1)^n$$

B. Prove the identities:

$$\left.\begin{array}{l} \sqrt{(F_{n+1}^2 - F_{2n+3}) \cdot (F_{n+3}^2 - F_{2n+7})} \\ \sqrt{(F_{n+3}^2 - F_{2n+5}) \cdot (F_{n+5}^2 - F_{2n+9})} \\ \sqrt{(F_{n+4}^2 - F_{2n+6}) \cdot (F_{n+6}^2 - F_{2n+10})} \end{array}\right\} = F_{n+2}F_{n+4} \quad \text{or} \quad F_{n+3}^2 + (-1)^n$$

SOLUTIONS

Woops!

The published solution to H-335, which appeared in the May 1983 issue of this quarterly is incorrect. The proposer (Paul Bruckman) pointed out that the polynomial in question can be factored as

$$(x-1)(x^2+bx-a^2)(x^2+ax-b^2),$$

where

$$a=(1+\sqrt{5})/2 \text{ and } b=(1-\sqrt{5})/2.$$

The desired roots may easily be obtained from this.

Old Timer

H-277 *Proposed by Larry Taylor, Rego Park, NY (Vol. 15, No. 4, December 1977)*

If $p \equiv \pm 1 \pmod{10}$ is prime and $x \equiv \sqrt{5}$ is of even order (mod p), prove that $x-3$, $x-2$, $x-1$, x, $x+1$, and $x+2$ are quadratic nonresidues of p if and only if $p \equiv 39 \pmod{40}$.

Solution by the proposer

Let $f \equiv (x+1)/2 \pmod{p}$. Then $f^3 \equiv x+2 \pmod{p}$. But $(f/p)=(f^3/p)$ and therefore

$$\left(\frac{x+1}{p}\right)\left(\frac{x+2}{p}\right)=(2/p).$$

In other words,

$$(2/p)=1 \tag{1}$$

is a necessary condition to have the six consecutive quadratic nonresidues of p.

Also, $f^2 \equiv (x+3)/2 \pmod{p}$. But $(2/p)=1$ has been established and, therefore,

$$\left(\frac{x+3}{p}\right)=1$$

Since $(x+3)(x-3) \equiv -4 \pmod{p}$, we have

$$\left(\frac{x+3}{p}\right)\left(\frac{x-3}{p}\right)=(-1/p).$$

But $\left(\frac{x-3}{p}\right)=-1$ is required and, therefore,

$$(-1/p)=-1 \tag{2}$$

is another necessary condition.

Since $(-1/p)=-1$ has been established and x is of even order (mod p), therefore $(x/p)=-1$. Since $2f^{-1} \equiv x-1 \pmod{p}$, therefore

$$\left(\frac{x+1}{p}\right)=\left(\frac{x-1}{p}\right).$$

Since $f^{-3} \equiv x - 2 \pmod{p}$, therefore

$$\left(\frac{x+2}{p}\right) = \left(\frac{x-2}{p}\right).$$

In [1], page 24, the following result is given:

$$(\sqrt{p}/5) = \left(\frac{-2x(x+1)}{p}\right).$$

Since $(2/p) = 1$, $(-1/p) = -1$, and $(x/p) = -1$ have been established, and

$$\left(\frac{x+1}{p}\right) = -1$$

is required, therefore

$$(\sqrt{p}/5) = -1 \tag{3}$$

is a third necessary condition.

There are, by inspection, no further necessary conditions. Therefore the logical product of (1), (2), and (3), which is equivalent to $p \equiv 39 \pmod{40}$, is a necessary and sufficient condition that $x - 3, x - 2, x - 1, x, x + 1$, and $x + 2$ are quadratic nonresidues of p.

Reference

1. Emma Lehmer. "Criteria for Cubic and Quartic Residuacity." *Mathematika* 5 (1958):20-29.

Late Acknowledgment: *E. Schmutz and P. Wittwer solved Problem H-333.*

Not in Prime Condition

H-345 *Proposed by Albert A. Mullin, Huntsville, AL (Vol. 20, No. 4, November 1982)*

Prove or disprove: No four consecutive Fibonacci numbers can be products of two distinct primes.

Solution by Lawrence Somer, Washington D.C.

The assertion is true. We, in fact, prove the following more general result:

Theorem: No three consecutive Fibonacci numbers can each be products of two distinct primes, except for the case

$$F_8 = 21 = 3 \cdot 7,\ F_9 = 34 = 2 \cdot 17,\ F_{10} = 55 = 5 \cdot 11.$$

Proof: We first show that a Fibonacci number F_n can be the product of exactly two distinct primes only if $n = 8$ or n is of the form p, $2p$, or p^2, where p is a prime. A prime p is a primitive divisor of F_n if $p \mid F_n$ but $p \nmid F_m$ for $0 < m < n$. R. Carmichael [1] proved that F_n has a primitive prime factor for every n except $n = 1, 2, 6,$ or 12. In none of these cases is F_n a product of exactly two distinct primes. It is also known that if $m|n$, then $F_m|F_n$. Thus, if n has two or more distinct proper divisors r and s that are not equal to 1, 2, 6, or 12, then F_n has at least three prime factors—the primitive prime factors of F_r, F_s, and F_{rs}, respectively. Since $F_6 = 8 = 2^3$, it follows that if n is a multiple of 6, then F_n is not a product of exactly two distinct prime factors. Since $F_1 = 1$ is not a product of two prime factors, it follows that if F_n is a product of two distinct prime factors, then F_n is of the form $F_{2^3} = F_8$, F_p, F_{2p}, or F_{p^2}, where p is a prime.

By inspection, one sees that if $n \leqslant 9$, then F_n, F_{n+1}, F_{n+2} are each products of two distinct primes only if $n = 8$. Now assume $n \geqslant 10$. Among the three consecutive integers n, $n + 1$, and $n + 2$, one of these numbers is divisible by 3. Call this number k. If the Fibonacci number F_k is the product of two distinct primes, then k is of the form p, $2p$, or p^2, where p is prime. This is impossible, since $3|k$ and $k > 9$. The theorem is now proved.

Reference

1. R. D. Carmichael. "On the Numerical Factors of the Arithmetic Forms $a^n + b^n$." *Annals of Mathematics* (2nd Ser.) 15 (1913):30-70.

Also solved by P. Bruckman and S. Singh.

Pell-Mell

H-346 *Proposed by Verner E. Hoggatt, Jr. (Deceased)*
(Vol. 20, No. 4, November 1982)

Prove or disprove: Let $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$ for $n = 1, 2, 3, \ldots$, then $P_7 = 169$ is the largest Pell number which is a square and there are no Pell numbers of the form $2s^2$ for $s > 1$.

Solution by M. Wachtel, Zurich, Switzerland

1.1 The roots of $x^2 - 2x - 1$ are $1 \pm \sqrt{2}$, and the quotient $\dfrac{P_{n+1}}{P_n} = 1 + \sqrt{2}$.

2.1 Pell numbers *with odd index* show the identity: $P_n^2 + P_{n+1}^2 = P_{2n+1}$.

2.2 P_{2n+1} will only be a square if

$$P_n^2 + P_{n+1}^2 = P_{2n+1} \quad \text{[see (2.1)]}$$

is identical with $(2m + 1)^2 + (2m^2 + 2m)^2 = (2m^2 + 2m + 1)^2$.

2.3 Obviously (1.1) and (2.2) will only be satisfied if $m = 2$ and $n = 3$, i.e., $5^2 + 12^2 = 169$. For $m > 2$, the quotient $2m^2 + 2m/2m + 1$ is rising, thus 169 is the greatest Pell number which is a square.

2.4 Using the general formula $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$, and setting $m - n = d$ (odd) yields:

$$(2dm + d^2)^2 + (2m^2 + 2dm)^2 = (2m^2 + 2dm + d^2)^2.$$

To satisfy (1.1), we have to set $m = 2d$, which leads to $5d^2 + 12d^2 = 13d^2$. It is easy to see that (2.2) will only be satisfied if $d = 1$, whereas setting $d = 3, 5, 7, \ldots$ will yield consecutive terms with common divisors, which is contrary to the Pell formula.

2.5 Pell numbers *with odd index* always are *odd* and can never be $2s^2$.

3.1 Pell numbers *with even index* show the *identity*:

$$2P_{n+1}(P_n + P_{n+1}) = P_{2n+2}.$$

3.2 Obviously, P_n and P_{n+1} are coprime, from which it follows that also P_{n+1} and $(P_n + P_{n+1})$ are coprime. Thus, P_{2n+2} can neither be a square nor twice a square.

Also solved by the proposer.

It All Adds Up

H-347 *Proposed by Paul S. Bruckman, Sacramento, CA (Vol. 20, No. 4, November 1982)*

Prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{x^n}{1 + x^{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{x^n}{(1 + (-x)^n)^2}, \text{ valid for all real } x \neq 0, \pm 1. \tag{1}$$

In particular, prove the identity:

$$\left\{\sum_{n=-\infty}^{\infty} \frac{1}{L_{2n}}\right\}^2 = \sum_{n=-\infty}^{\infty} \frac{1}{L_n^2} \tag{2}$$

Solution by the proposer

Let

$$f(x) \equiv \sum_{n=-\infty}^{\infty} x^{n^2}, \text{ where } -1 < x < 1. \tag{3}$$

Theorems 311 and 312 in [1] state (using our notation):

$$(f(x))^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{1 - x^{2n-1}}; \tag{4}$$

while Theorem 385 in [1] states:

$$(f(x))^2 = 1 + 8\sum \frac{mx^m}{1 - x^m}, \text{ where } m \text{ runs through all positive integral values which are not multiples of 4.} \tag{5}$$

We may rearrange the terms in (4) and (5), since the series are absolutely convergent for $|x| < 1$. Thus,

$$(f(x))^2 = 1 + 4\sum_{m,\,n=1}^{\infty}(-1)^{n-1}x^{(2n-1)m} = 1 + 4\sum_{m=1}^{\infty}x^m\sum_{n=0}^{\infty}(-x^{2m})^n$$

$$= 1 + 4\sum_{m=1}^{\infty}\frac{x^m}{1 + x^{2m}}.$$

If

$$u_m(x) = \frac{x^m}{1 + x^{2m}},$$

we note that for all $x \neq 0$,

$$u_0(x) = 1/2 \quad \text{and} \quad u_m(x) = u_{-m}(x) = u_m(1/x).$$

Therefore, the *transformed* series for $(f(x))^2$ converges for all real $x \neq 0, \pm 1$. It follows that

$$(f(x))^2 = 2\sum_{n=-\infty}^{\infty}\frac{x^n}{1 + x^{2n}}, \text{ for all real } x \neq 0, \pm 1. \tag{6}$$

By similar reasoning,

$$\sum_{n=1}^{\infty}\frac{nx^n}{1 - x^n} = \sum_{m,\,n=1}^{\infty}nx^{mn} = \sum_{m=1}^{\infty}x^m\sum_{n=0}^{\infty}(n + 1)x^{mn} = \sum_{m=1}^{\infty}\frac{x^m}{(1 - x^m)^2}.$$

Therefore, using (5),

$$(f(x))^4 = 1 + 8\sum_{n=1}^{\infty}\frac{nx^n}{1 - x^n} - 32\sum_{n=1}^{\infty}\frac{nx^{4n}}{1 - x^{4n}}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^n}{(1 - x^n)^2} - 32\sum_{n=1}^{\infty}\frac{x^{4n}}{(1 - x^{4n})^2}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1 - x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1 - x^{2n})^2} - 32\sum_{n=1}^{\infty}\frac{x^{4n}}{(1 - x^{4n})^2}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1 - x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1 - x^{4n})^2}\left\{(1 + x^{2n})^2 - 4x^{2n}\right\}$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1 - x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1 - x^{4n})^2}(1 - x^{2n})^2$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(1 - x^{2n-1})^2} + 8\sum_{n=1}^{\infty}\frac{x^{2n}}{(1 + x^{2n})^2}$$

$$= 1 + 8\sum_{m=1}^{\infty}\frac{x^m}{(1 + (-x)^m)^2}.$$

If

$$v_m(x) = \frac{x^m}{(1 - (-x)^m)^2},$$

we note that for all $x \neq 0$,

$$v_0(x) = 1/4 \quad \text{and} \quad v_m(x) = v_{-m}(x) = v_m(1/x).$$

As before, the transformed series for $(f(x))^4$ must therefore converge for all real $x \neq 0, \pm 1$. We then see that

$$(f(x))^4 = 4 \sum_{n=-\infty}^{\infty} \frac{x^n}{(1 + (-x)^n)^2}, \text{ for all real } x = 0, \pm 1. \tag{7}$$

Squaring both sides of (6) and comparing with (7) yields (1). As a special case, we set $x = b^2$, where $b = (1/2)(1 - \sqrt{5})$, and obtain (2).

It should be pointed out that (2) was derived in [2] and given there as relation (59), using elliptic function theory. Indeed, relations (4) and (5) above have their basis in elliptic function theory.

References

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers.* 4th ed. Oxford: Clarendon Press, 1960; rpt. (with corrections), 1962.
2. Paul S. Bruckman. "On the Evaluation of Certain Infinite Series by Elliptic Functions." *The Fibonacci Quarterly* 15 (1977):293-310.

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