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PARTITIONS, COMPOSITIONS AND CYCLOMATIC NUMBER OF FUNCTION LATTICES

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1. INTRODUCTION

By a poset we mean a partially ordered set. If G, H are posets, then their **cardinal power** G^H is defined by Birkhoff (see [1], p. 55) as a set of all order-preserving mappings of the poset H into the poset G with an ordering defined as follows.

For f, $g \in G^H$ there holds $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for every $x \in H$.

If G is a lattice, then G^H is usually called a **function lattice.** (It is easy to prove that if G is a lattice, or modular lattice, or distributive one, then so is G^H (see [1], p. 56).

Let A be a poset and let a, $b \in A$ with $a < b$. If no $x \in A$ exists such that $a < x < b$, then b is said to be a **successor** of the element a. Let $n(a)$ denote the number of all the successors of the element $a \in A$. Further, let $c(A)$ denote the number of all the components of the poset A, i.e., the number of its maximal continuous subsets.

Finally, if X is a set, then $|X|$ is its cardinal number.

Now we can introduce the following definition.

Definition: Let $A \neq \phi$ be a finite poset. Put

(a) $$n(A) = \sum_{a \in A} n(a) \tag{1.1}$$

(b) $$d(A) = \frac{n(A)}{|A|} \tag{1.2}$$

(c) $$v(A) = n(A) - |A| + c(A) \tag{1.3}$$

The number $d(A)$ is called the **density** of the poset A, the number $v(A)$ is called the **cyclomatic number** of the poset A.

It is evident that, for a finite poset A, $n(A)$ is equal to the number of edges in the Hasse diagram of the poset A, thus $v(A)$ is the cyclomatic number of the mentioned Hasse diagram in the sense of graph theory.

Now our aim is to determine the density and the cyclomatic number of functional lattices G^H, where G, H are finite chains.

2. PARTITIONS AND COMPOSITIONS

The symbols N, N_0, will always denote, respectively, the positive integers, the nonnegative integers.

Let $k, n, s \in N$. By a **partition** of n into k summands, we mean, as usual, a k-tuple $a_1, a_2, \ldots, a_k$ such that each $a_i \in N$, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k$, and

$$a_1 + \cdots + a_k = n. \tag{2.1}$$

Let $P(n, k)$ denote the set of all the partitions of the number n into k summands. Let $P(n, k, s)$ denote the set of those partitions of n into k summands, in which the summands are not greater than the number s, i.e., such that $s \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k \geqslant 1$.

By a **composition** of the number n into k summands, we mean an ordered k-tuple $(a_1, \ldots, a_k)$, with $a_i \in N$, satisfying (2.1). Let $C(n, k)$ denote the set of all these compositions.

Finally, let $D(n, k)$ denote the set of all the compositions of the number n into k summands $a_i \in N_0$ [so that $C(n, k) \subseteq D(n, k)$].

It is easy to determine the number of elements of the sets $P(n, k)$, $P(n, k, s)$, $C(n, k)$, and $D(n, k)$—see, e.g., [2], [3], and [6].

Theorem 1: For $k, n, s \in N$,

(a) $$|P(n, k)| = \sum_{i=1}^{k} |P(n - k, i)| \tag{2.2}$$

(b) $$|P(n, k, s + 1)| = \sum_{i=1}^{k} |P(n - k, i, s)| \tag{2.3}$$

(c) $$|C(n, k)| = \binom{n - 1}{k - 1} \tag{2.4}$$

(d) $$|D(n, k)| = \binom{n + k - 1}{k - 1} \tag{2.5}$$

Definition: Define the binary relation ρ on the set $D(n, k)$ as follows:

If $\alpha, \beta \in D(n, k)$, $\alpha = (a_1, \ldots, a_k)$, $\beta = (b_1, \ldots, b_k)$, then $\alpha\rho\beta$ if and only if $i \in \{2, 3, \ldots, k\}$ exists such that

$$b_{i-1} = a_{i-1} + 1,\ b_i = a_i - 1, \text{ and } b_j = a_j \text{ for remaining } j.$$

Further, put, for $\alpha \in D(n, k)$,

$$\Gamma(\alpha) = \{\beta \in D(n, k);\ \alpha\rho\beta\}.$$

Remark: Thus, for $\alpha = (a_1, \ldots, a_k) \in D(n, k)$, the elements from $\Gamma(\alpha)$ are all the compositions of the form

$$(a_1, \ldots, a_{i-1} + 1, a_i - 1, a_{i+1}, \ldots, a_k).$$

From the definitions of the set $D(n, k)$ and the relation ρ, it follows that $|\Gamma(\alpha)|$ is equal to the number of nonzero summands a_i, $i = 2, \ldots, k$, in α.

Definition: For $i \in N_0$, we denote

$$D^i(n, k) = \{\alpha \in D(n, k);\ |\Gamma(\alpha)| = i\}. \tag{2.6}$$

Theorem 2: For $k, n \in N$, $i \in N_0$,

$$|D^i(n, k)| = \binom{n}{i}\binom{k-1}{i}. \tag{2.7}$$

Proof: Let $\alpha = (a_1, \ldots, a_k) \in D^i(n, k)$. Therefore, according to the above Remark, there are only i numbers that are nonzeros from all the summands $a_2, \ldots, a_k$, a_1 being arbitrary. If now $a_1 = j$, then by (2.4) there exist precisely

$$\binom{k-1}{i}\binom{n-j-1}{i-1}$$

compositions of the required form. Hence,

$$|D^i(n, k)| = \binom{k-1}{i}\left[\binom{n-1}{i-1} + \binom{n-2}{i-1} + \cdots + \binom{i-1}{i-1}\right] = \binom{k-1}{i}\binom{n}{i}.$$

Remark: It is evident that

$$D^i(n, k) \neq \phi \text{ if and only if } i \leqslant \min[k-1, n].$$

Let Y denote the set of all nonincreasing sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of nonnegative integers in which there are only finitely many $a_i \neq 0$, i.e., such that

$$\sum_{i=1}^{\infty} a_i < \infty.$$

We define the ordering $\leqslant$ on the set Y by:

$$(a_1, a_2, \ldots) \leqslant (b_1, b_2, \ldots) \text{ if and only if } a_i \leqslant b_i \text{ for every } i \in N.$$

Then the poset Y is evidently a distributive lattice. It is the so-called **Young lattice**. For more details on its properties see, e.g., [4] and [5].

Identifying $(a_1, \ldots, a_k) \in P(n, k)$ with $(a_1, \ldots, a_k, 0, 0, \ldots) \in Y$, we henceforth consider the partitions as elements of the Young lattice.

The elements with a height n in Y are evidently all the partitions of the number n. [The element with the height 0 is obviously the sequence

$(0, 0, \ldots)$].

Definition: For $\alpha \in Y$, the **principal ideal** $Y(\alpha)$ is given by

$$Y(\alpha) = \{\beta \in Y;\ \beta \leqslant \alpha\}.$$

Definition: Let σ denote the **covering relation** on the lattice Y, i.e., for $\alpha, \beta \in Y$,

$\alpha\sigma\beta$ if and only if β is a successor of the element α.

The next result follows immediately from the definition of the Young lattice.

Theorem 3: Let $\alpha, \beta \in Y$, $\alpha = (a_1, a_2, \ldots)$, $\beta = (b_1, b_2, \ldots)$. Then $\alpha\sigma\beta$ if and only if there exists $i \in N$ such that $b_i = a_i + 1$, and $a_j = b_j$ for $j \in N$, $j \neq i$.

Definition: Let $\alpha = (a_1, a_2, \ldots) \in Y$, let r be the number $a_i \in \alpha$, with $a_i \neq 0$. Then **canonical mapping** $f : Y(\alpha) \to D(r, 1 + a_1)$ is defined as follows:

For $\beta = (b_1, b_2, \ldots) \in Y(\alpha)$, the image $f(\beta)$ is the composition

$$(c_1, c_2, \ldots, c_{1+a_1}) \in D(r, 1 + a_1)$$

for which c_i is the number of values of j for which $b_j = a_1 + 1 - i$ in the r-tuple $(b_1, \ldots, b_r)$.

Remark: If $\beta = (b_1, \ldots, b_r, 0, 0, \ldots) \in Y(\alpha)$, then evidently $0 \leqslant b_i \leqslant a_i$ for all $i = 1, \ldots, r$. The image of the sequence β under the canonical mapping f is the composition $(c_1, \ldots, c_{1+a_1})$ with the following properties:

c_1 is the number of integers a_1 in $(b_1, \ldots, b_r)$, c_2 is the number of values of j for which $b_j = a_1 - 1$ in $(b_1, \ldots, b_r)$, etc., until c_{1+a_1} is the number of zeros in $(b_1, \ldots, b_r)$.

Theorem 4: Let $\alpha = (a_1, \ldots, a_r, 0, 0, \ldots) \in Y$, with $a_1 = \cdots = a_r = k > 0$. Then

$$(Y(\alpha), \sigma) \cong [D(r, k + 1), \rho]. \tag{2.8}$$

Proof: Let $f : Y(\alpha) \to D(r, k + 1)$ be the canonical mapping. Then f is evidently a bijection. Let $\beta, \gamma \in Y(\alpha)$. If $\beta = (b_1, b_2, \ldots)$ and if $\beta\sigma\gamma$, then by Theorem 3, $\gamma = (b_1, \ldots, b_i + 1, b_{i+1}, \ldots)$ for some i. Denote b_i by t. Then there is in the sequence γ one less t and one more $t + 1$ than in the sequence β. Combining this fact with the definition of the relation ρ on $D(r, k + 1)$, we have

$$\beta\sigma\gamma \text{ if and only if } f(\beta)\rho f(\gamma). \tag{2.9}$$

Thus, the canonical mapping f is an isomorphism from $(Y(\alpha), \sigma)$ on

$$[D(r, k + 1), \rho].$$

3. DENSITY AND CYCLOMATIC NUMBER OF FUNCTION LATTICES

Let P, Q be arbitrary posets. If $P = \phi$, $Q \neq \phi$, then $P^Q = \phi$. If $Q = \phi$, then $P^Q = \{\phi\}$, P being arbitrary. Henceforth, we shall consider only such functional lattices P^Q, where $P \neq \phi \neq Q$.

The basic properties of the functional lattices P^Q, where P, Q are finite chains, are described in [6]. Namely, there holds

Theorem 5: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

(a) $$|P^Q| = \binom{p + q - 1}{q} \tag{3.1}$$

(b) For $i \in N_0$, the number of elements in P^Q with height i, is equal to $|P(q + i, q, p)|$.

(c) $P^Q \cong Y(\alpha)$, where $\alpha = (a_1, \ldots, a_q, 0, 0, \ldots)$, $a_1 = \cdots = a_q = p - 1$.

Proof: The assertion (a) is trivial. The proof of the assertion (b) is in [6], p. 9. The assertion (c) results from the following: Put

$$P = \{0 < 1 < \cdots < p - 1\},\ Q = \{1 < 2 < \cdots < q\}.$$

The isomorphism $F : P^Q \to Y(\alpha)$ is given by

$$F(f) = (f(q), f(q - 1), \ldots, f(1), 0, 0, \ldots),$$

for each $f \in P^Q$.

Lemma: For $k, n \in N$,

(a) $$\sum_{i=0}^{n} \binom{k}{i}\binom{n}{i} = \binom{k + n}{n} \tag{3.2}$$

(b) $$\sum_{i=0}^{n} i\binom{k}{i}\binom{n}{i} = \frac{kn}{k + n}\binom{k + n}{n} \tag{3.3}$$

Proof: (a) The assertion (3.2) is well known.

(b) In [8], Hagen states without proofs many combinatorial identities. As the 17th there is stated:

$$\sum_{i=0}^{n} \frac{a + bi}{(p - id)(q + id)}\binom{p - id}{n - i}\binom{q + id}{i} = \frac{a(p + q - ni) + bnq}{q(p + q)(p - id)}\binom{p + q}{n}. \tag{3.4}$$

The first very complicated proof of formula (3.4) was given by Jensen in 1902. The simplest of the known proofs is given in [9].

Substituting $a = 0$, $b = 1$, $p = n$, $q = k$, $d = 0$ into (3.4), we obtain

$$\sum_{i=0}^{n} \frac{i}{kn}\binom{n}{n - i}\binom{k}{i} = \frac{kn}{(k + n)kn}\binom{k + n}{n},$$

by which formula (3.3) is proved.

Theorem 6: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

$$n(P^Q) = \frac{q(p - 1)}{p + q - 1} \cdot \binom{p + q - 1}{q}. \tag{3.5}$$

Proof: If $p = 1$, then $|P^Q| = 1$ so that $n(P^Q) = 0$ and (3.5) is evidently valid. Thus let $p > 1$. By Theorem 5(c), we have $P^Q \cong Y(\alpha)$, where

$\alpha = (a_1, \ldots, a_q, 0, 0, \ldots)$, $a_1 = \cdots = a_q = p - 1$. Let $f: Y(\alpha) \to D(p, q)$ be the canonical mapping. For $\beta \in Y(\alpha)$, $n(\beta) = |\Gamma[f(\beta)]|$ by Theorem 4. Combining this fact with (2.7) and (3.3), we obtain

$$n(P^Q) = \sum_{\beta \in Y(\alpha)} n(\beta) = \sum_{i=0}^{q} i\binom{q}{i}\binom{p-1}{i} = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{i}.$$

Remark: Combining (3.1), (3.5), and the proof of Theorem 6, we have

$$n(P^Q) = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{q} = \frac{q(p-1)}{p+q-1} \cdot |P^Q| = \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i}. \quad (3.6)$$

Now it is easy to determine the density and also the cyclomatic number of the functional lattice P^Q.

Theorem 7: Let $p, q \in N$, let P, Q be chains such that $|P| = p$, $|Q| = q$. Then

(a) $$d(P^Q) = \frac{q(p-1)}{p+q-1} \quad (3.7)$$

(b) $$\nu(P^Q) = \sum_{i=1}^{q} (i-1)\binom{q}{i}\binom{p-1}{i} \quad (3.8)$$

Proof: (a) The assertion (3.7) follows from (1.2) and (3.6).

(b) If A is a connected poset, then $c(A) = 1$. Combining this fact with (1.3), (3.6), (3.1), and (3.2), we obtain

$$\begin{aligned}\nu(P^Q) &= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \binom{p+q-1}{q} + 1\\ &= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \sum_{i=0}^{q}\binom{q}{i}\binom{p-1}{i} + 1\\ &= \sum_{i=1}^{q} i\binom{q}{i}\binom{p-1}{i} - \sum_{i=1}^{q}\binom{q}{i}\binom{p-1}{i} = \sum_{i=1}^{q}(i-1)\binom{q}{i}\binom{p-1}{i}.\end{aligned}$$

Remark: Combining (3.6), (3.8), and (3.1), we obtain

$$\nu(P^Q) = \sum_{i=1}^{q}(i-1)\binom{q}{i}\binom{p-1}{i} = \frac{q(p-1)}{p+q-1}\binom{p+q-1}{q} - \binom{p+q-1}{q} + 1. \quad (3.9)$$

Let $p, q, r, s \in N$, and let P, Q, R, S be chains such that $|P| = p$, $|Q| = q$, $|R| = r$, $|S| = s$. By (3.7) and (3.8),

if $r = q + 1$, $s = p - 1$, then $d(P^Q) = d(R^S)$, $\nu(P^Q) = \nu(R^S)$. (3.10)

But in [7] we have proved that for $p > 1$,

$$P^Q \cong R^S \text{ if and only if } p = r,\ q = s \text{ or } r = q + 1,\ s = p - 1.$$

(3.10) is now evident.

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