LOWER BOUNDS FOR UNITARY MULTIPERFECT NUMBERS

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1. INTRODUCTION

Throughout this paper n and k will denote positive integers that exceed 2. With or without a subscript, p will denote a prime, and the i^{th} odd prime will be symbolized by P_i. If d is a positive integer such that $d|n$ and $(d, n/d) = 1$, then d is said to be a unitary divisor of n. The sum of all of the unitary divisors of n is symbolized by $\sigma^*(n)$. If $n = p_1^{a_1}p_2^{a_2} \ldots p_s^{a_s}$, where the p_i are distinct and $a_i > 0$ for all i, then it is easy to see that

$$\sigma^*(n) = \prod_{i=1}^{s}(1 + p_i^{a_i}) \tag{1}$$

and that σ^* is a multiplicative function.

Subbarao and Warren [2] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$. Five unitary perfect numbers have been found (see [3]). The smallest is 6, the largest has 24 digits. Harris and Subbarao [1] have defined n to be a *unitary multiperfect number* (UMP) if $\sigma^*(n) = kn$, where $k > 2$. We know of no example of a unitary multiperfect number and, as we shall see, if one exists it must be very large.

Suppose first that $n = p_1^{a_1}p_2^{a_2} \ldots p_s^{a_s}$, where n is odd and $\sigma^*(n) = kn$. Assume that $k = 2^c M$, where $2 \nmid M$ and $c \geqslant 0$. Then, since

$$2|(1 + p_i^{a_i}) \text{ for } i = 1, 2, \ldots, s,$$

it follows from (1) that $s \leqslant c$. Also,

$$2^c M = k = \sigma^*(n)/n = \prod_{i=1}^{s}(1 + p_i^{-a_i}) < 2^s \leqslant 2^c,$$

which is a contradiction. We have proved

Theorem 1: There are no odd unitary multiperfect numbers.

This result was stated in [1]. Its proof is included here for the sake of completeness.

2. LOWER BOUNDS FOR UNITARY MULTIPERFECT NUMBERS

We assume from now on that

$$n = 2^\alpha \prod_{i=1}^{t} p_i^{a_i}, \text{ where } \alpha a_i > 0 \text{ and } 3 \leqslant p_1 < p_2 < \cdots < p_t. \tag{2}$$

Also, $\sigma^*(n) = kn$, so that

$$k = \sigma^*(n)/n = (1 + 2^{-\alpha})\prod_{i=1}^{t}(1 + p_i^{-a_i}). \tag{3}$$

Since $2 \mid (1 + p_i^{\alpha_i})$, it follows from (1) and (2) that $t \leqslant \alpha + 2$ if $k = 4$, and $t \leqslant \alpha + 1$ if $k = 6$. Therefore, since $1 + x^{-1}$ is monotonic decreasing for $x > 0$, it follows from (3), if $k = 4$ or 6, that

$$4 \leqslant k \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{t} (1 + P_i^{-1}) \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha+2} (1 + P_i^{-1}) = F(\alpha).$$

A computer run showed that $F(\alpha) < 4$ for $\alpha \leqslant 48$. Therefore, $\alpha \geqslant 49$ if $k = 4$ or 6. Also, from (3),

$$4 \leqslant k \leqslant (1 + 2^{-49}) \prod_{i=1}^{t} (1 + P_i^{-1}) = G(t).$$

Since $G(50) < 4$, we see that $t \geqslant 51$. Thus

$$n \geqslant 2^{49} \prod_{i=1}^{51} P_i > 10^{110} \text{ if } k = 4 \text{ or } 6.$$

If $k \geqslant 8$, then

$$8 \leqslant k \leqslant 1.5 \prod_{i=1}^{t} (1 + P_i^{-1}) = H(t).$$

A computer run showed that $H(t) < 8$ for $t \leqslant 246$. Therefore, if $k \geqslant 8$, $t \geqslant 247$ and

$$n \geqslant 2 \prod_{i=1}^{247} P_i > 10^{663}.$$

Now suppose that k is odd and $k \geqslant 5$. Since $2 \mid (1 + p_i^{\alpha_i})$, we see that $t \leqslant \alpha$. Also, from (3),

$$5 \leqslant k \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha} (1 + P_i^{-1}) = J(\alpha);$$

and since $J(\alpha) < 5$ for $\alpha \leqslant 165$, it follows that $\alpha \geqslant 166$. Moreover,

$$5 \leqslant k \leqslant (1 + 2^{-166}) \prod_{i=1}^{t} (1 + P_i^{-1}) = K(t),$$

and since $K(165) < 5$, we see that $t \geqslant 166$. Therefore, if $k \geqslant 5$ and k is odd, then

$$n \geqslant 2^{166} \prod_{i=1}^{166} P_i > 10^{461}.$$

Theorem 2: Suppose that n is a UMP with t distinct odd prime factors and that $\sigma^*(n) = kn$. If $k \geqslant 8$, then $n > 10^{663}$ and $t \geqslant 247$. If $k = 4$ or 6, then $n > 10^{110}$, $t \geqslant 51$, and $2^{49} \mid n$. If k is odd and $k \geqslant 5$, then $n > 10^{461}$, $t \geqslant 166$, and $2^{166} \mid n$.

3. UNITARY TRIPERFECT NUMBERS

If $\sigma^*(n) = 3n$, n will be said to be a *unitary triperfect number*. Throughout this section we assume that n is such a number. We shall denote by q_i the i^{th} prime congruent to 1 modulo 3 and by Q_i the i^{th} prime congruent to 2 modulo 3. If $3 \nmid n$, then $t \leqslant \alpha$ and, from (3),

$$3 \leqslant (1 + 2^{-\alpha}) \prod_{i=2}^{\alpha+1} (1 + P_i^{-1}) = L(\alpha).$$

Since $L(\alpha) < 3$ for $\alpha \leqslant 49$, we see that $\alpha \geqslant 50$. Also,

$$3 \leqslant (1 + 2^{-50}) \prod_{i=2}^{t+1} (1 + P_i^{-1}) = M(t),$$

and since $M(49) < 3$, it follows that $t \geqslant 50$. And, finally, since $3 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$, we see that

$$n \geqslant 2^{50} 5^2 11^2 17^2 23 \prod_{i=1}^{46} q_i > 10^{105}. \quad \text{(Note that } q_{46} = 523.)$$

If $3 \| n$, then $t \leqslant \alpha - 1$, since

$$3 \cdot 2^\alpha \prod_{i=1}^{t} p_i^{a_i} = (1 + 2^\alpha)(4) \prod_{i=2}^{t} (1 + p_i^{a_i}).$$

From (3),

$$3 = (1 + 2^{-\alpha})(4/3) \prod_{i=2}^{t} (1 + p_i^{-a_i}) \leqslant (1 + 2^{-\alpha}) \prod_{i=1}^{\alpha-1} (1 + P_i^{-1}) = N(\alpha),$$

and since $N(\alpha) < 3$ for $\alpha \leqslant 16$, we see that $\alpha \geqslant 17$. Also, $3^2 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$. Therefore, since $1 + x^{-1}$ is monotonic decreasing for $x > 0$, *and* since

$$(1 + 2^{-17})(4/3)(6/5)(12/11)(290/17^2) \prod_{i=1}^{40} (1 + q_i^{-1}) < 3,$$

it follows from (3) that $t \geqslant 45$. Thus, $\alpha \geqslant 46$ and

$$n \geqslant 2^{46} 3 \cdot 5 \cdot 11 \cdot 17^2 \prod_{i=1}^{41} q_i > 10^{107}. \quad \text{(Note that } q_{41} = 439.)$$

If $3^2 \| n$, then $t \leqslant \alpha$ and, from (3),

$$3 \leqslant (1 + 2^{-\alpha})(10/9) \prod_{i=2}^{\alpha} (1 + P^{-1}) = R(\alpha).$$

$\alpha \geqslant 32$, since $R(\alpha) < 3$ for $\alpha \leqslant 31$. Also, $3^3 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2$ (mod 3). Therefore, since

$$(1 + 2^{-32})(10/9)(6/5)(12/11)(24/23)(290/17^2) \prod_{j=5}^{8} (1 + Q_j^{-2}) \prod_{i=1}^{227} (1 + q_i^{-1}) < 3,$$

we see that $t \geqslant 237$. ($Q_8 = 53$ and $q_{227} = 3307$.) Thus, $\alpha \geqslant 237$ and

$$n \geqslant 2^{237}(5 \cdot 11 \cdot 23)(3 \cdot 17 \cdot 29 \cdot 41 \cdot 47 \cdot 53)^2 \prod_{i=1}^{228} q_i > 10^{779}.$$

If $3^3 \| n$, then $t \leqslant \alpha - 1$ and

$$3 \leqslant (1 + 2^{-\alpha})(28/27) \prod_{i=2}^{\alpha-1} (1 + P_i^{-1}) = S(\alpha).$$

Since $S(\alpha) < 3$ for $\alpha \leqslant 43$, we see that $\alpha \geqslant 44$. Also, $3^4 \| \sigma^*(n)$ and $3 | (1 + p)$ if $p \equiv 2 \pmod 3$. Therefore, since

$$(1 + 2^{-44})(28/27)(6/5)(12/11)(18/17)\prod_{j=4}^{12}(1 + Q_j^{-2})\prod_{i=1}^{530}(1 + q_i^{-1}) < 3,$$

we conclude that $t \geqslant 544$. ($Q_{12} = 89$ and $q_{530} = 8623$.) Thus, $\alpha \geqslant 545$ and

$$n \geqslant 2^{545}3^3 \cdot 5 \cdot 11 \cdot 17 \prod_{j=4}^{12} Q_j^2 \prod_{i=1}^{531} q_i > 10^{2026}.$$

If $3^4 | n$, then $t \leqslant \alpha$ and

$$3 \leqslant (1 + 2^{-\alpha})(82/81)\prod_{i=2}^{\alpha}(1 + P_i^{-1}) = T(\alpha).$$

Since $T(\alpha) < 3$ for $\alpha \leqslant 47$, it follows that $\alpha \geqslant 48$. From (3),

$$3 \leqslant (1 + 2^{-48})(82/81)\prod_{i=2}^{t}(1 + P_i^{-1}) = U(t),$$

and since $U(47) < 3$, we conclude that $t \geqslant 48$ and

$$n \geqslant 2^{48}3^4 \prod_{i=2}^{48} P_i > 10^{102}.$$

We summarize these results in the following theorem.

Theorem 3: Suppose that n is a unitary triperfect number with t distinct odd prime factors. Then $t \geqslant 45$, $n > 10^{102}$, and $2^{46} | n$. If $3^2 \| n$, then $t \geqslant 237$, $n > 10^{779}$, and $2^{237} | n$. If $3^3 \| n$, then $t \geqslant 544$, $n > 10^{2026}$, and $2^{545} | n$.

REFERENCES

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