THE GENERATION OF HIGHER-ORDER LINEAR RECURRENCES FROM SECOND-ORDER LINEAR RECURRENCES

LAWRENCE SOMER
1400 20th St., NW #619, Washington, D.C. 20036
(Submitted May 1981)

Let $\{u_n\}$ be a Lucas sequence of the first kind defined by the second-order recursion relation

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are integers and $u_0 = 0$, $u_1 = 1$. By the Binet formulas

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta),$$

where α and β are roots of the characteristic polynomial

$$x^2 - ax - b.$$

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of the characteristic polynomial of $\{u_n\}$. We shall prove the following theorem which demonstrates that the quotients of specified terms of the second-order recurrence $\{u_n\}$ satisfy a higher-order relation.

Theorem 1: Consider the sequence

$$\{w_n\}_{n=1}^{\infty} = \{u_{nk}/u_n\}_{n=1}^{\infty},$$

where k is a fixed positive integer, $\alpha\beta \neq 0$, and α/β is not a root of unity. Then $\{w_n\}$ satisfies a k^{th}-order linear integral recursion relation. Further, the order k is minimal.

Along the lines of this theorem, Selmer [1] has shown how one can form a higher-order linear recurrence consisting of the term-wise products of two other linear recurrences. In particular, let $\{s_n\}$ be an m^{th}-order and $\{t_n\}$ be a p^{th}-order linear integral recurrence with the associated polynomials $s(x)$ and $t(x)$, respectively. Let α_i, $i = 1, 2, \ldots, m$, and β_j, $j = 1, 2, \ldots, p$, be the roots of the polynomials $s(x)$ and $t(x)$, respectively, and assume that each polynomial has no repeated roots. Then, the sequence

$$\{h_n\} = \{s_n t_n\}$$

satisfies a linear integral recurrence of order at most mp, whose characteristic polynomial $h(x)$ has roots consisting of the r distinct elements of the set $\{\alpha_i\beta_j\}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$. Note that the coefficients of $h(x)$ are integral because they are symmetric in the conjugate algebraic integers $\alpha_i\beta_j$. However, $\{h_n\}$ may satisfy a recursion relation of order lower than r.

Selmer's proof depends on the fact that the recurrences $\{s_n\}$ and $\{t_n\}$ can be expressed in terms of their characteristic roots by means of the formulas

$$s_n = \sum_{i=1}^{m} \gamma_i \alpha_i^n, \quad t_n = \sum_{j=1}^{p} \delta_j \beta_j^n. \tag{1}$$

This follows from the fact that the sequences $\{\alpha_i^n\}$, $1 \leqslant i \leqslant m$, and $\{\beta_j^n\}$, $1 \leqslant j \leqslant p$, satisfy the same recursion relations as $\{s_n\}$ and $\{t_n\}$, respectively. Further, a linear combination of sequences satisfying the same linear recursion relation also satisfies that linear recursion relation. By means of Cramer's rule, one can then solve (1) for s_n, $1 \leqslant n \leqslant m$, and t_n, $1 \leqslant n \leqslant p$, in terms of α_i^n, $1 \leqslant i \leqslant m$, and β_j^n, $1 \leqslant j \leqslant p$, respectively. The fact that the roots α_i, $1 \leqslant i \leqslant m$, and β_j, $1 \leqslant j \leqslant p$, are distinct guarantees unique solutions in terms of α_i^n and β_j^n. Now,

$$h_n = s_n t_n = \left(\sum_{i=1}^{m} \gamma_i \alpha_i^n\right)\left(\sum_{j=1}^{p} \delta_j \beta_j^n\right) = \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant p}} \gamma_i \delta_j (\alpha_i \beta_j)^n,$$

and each $\alpha_i \beta_j$ is a root of the polynomial $h(x)$.

Before proving our main result, we will need the following lemma. A proof of this lemma is given by Willett [2].

Lemma 1: Consider the sequence $\{s_n\}$. We introduce the determinant

$$D_r(t) = \begin{vmatrix} s_t & s_{t+1} & \cdots & s_{t+r-1} \\ s_{t+1} & s_{t+2} & & s_{t+r} \\ \cdots & \cdots & \cdots & \cdots \\ s_{t+r-1} & s_{t+r} & & s_{t+2r-2} \end{vmatrix}$$

Then $\{s_n\}$ satisfies a recursion relation of minimal order k if and only if

$$D_k(0) \neq 0$$

and

$$D_r(0) = 0 \text{ for } r > k.$$

We are now ready for the proof of the main result. The first part of the proof will show that $\{w_n\}$ satisfies a k^{th}-order linear integral recursion relation. The second part of the proof will establish the minimality of k. The simple proof of minimality was suggested by Professor Ernst S. Selmer.

Proof of Theorem 1: First, we claim that $u_n \neq 0$ for $n \geqslant 1$ and $\{w_n\}$ is well-defined. If $u_n = 0$, then $\alpha^n - \beta^n = 0$ and $(\alpha/\beta)^n = 1$, since $\beta \neq 0$. This is impossible because α/β is not a root of unity. Note that

$$w_n = \sum_{i=0}^{k-1} \alpha^{(k-1-i)n} \cdot \beta^{in}.$$

The k algebraic integers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, are all distinct because α/β is not a root of unity. If α and β are rational integers, then the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, certainly satisfy a monic polynomial of degree k over the rational integers. If α and β are irrational, then α and β are conjugate in the algebraic number field $K = Q(\alpha, \beta) = Q(\alpha)$, where Q denotes the rational numbers. Thus, $\alpha^{k-1-i}\beta^i$ and $\alpha^i\beta^{k-1-i}$ are conjugate in K. Hence, the numbers $\alpha^{k-1-i}\beta^i$, $0 \leqslant i \leqslant k-1$, satisfy a polynomial of degree k which is a product of monic, integral quadratic polynomials and at most one monic, integral linear polynomial. By our discussion preceding the statement of Lemma 1, the sequences $\{(\alpha^{k-1-i}\beta^i)^n\}_{n=1}^{\infty}$, $0 \leqslant i \leqslant k-1$, all satisfy the same linear integral recursion relation of order k. Thus, $\{w_n\}_{n=1}^{\infty}$, the sum of these k sequences, also satisfies this same recursion relation.

To prove the minimality of k, we first note that $\{w_n\}$ may also be defined for $n = 0$ if we put $w_0 = k$. Replacing $D_r(t)$ of Lemma 1 by $D_r(s_n, t)$, the minimality will follow if we can show that $D_k(w_n, 0) \neq 0$. To illustrate the method, let us take $k = 3$ as an example, when

$$D_k(w_n,\ 0) = \begin{vmatrix} 3 & \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 \\ \alpha^2 + \alpha\beta + \beta^2 & \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 \\ \alpha^4 + \alpha^2\beta^2 + \beta^4 & \alpha^6 + \alpha^3\beta^3 + \beta^6 & \alpha^8 + \alpha^4\beta^4 + \beta^8 \end{vmatrix}.$$

The corresponding matrix may be written as the product

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha^2 & \alpha\beta & \beta^2 \\ \alpha^4 & \alpha^2\beta^2 & \beta^4 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{pmatrix}$$

Thus, $D_k(w_n, 0)$ is the square of a Vandermonde determinant:

$$D_k(w_n,\ 0) = \begin{vmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha\beta & \alpha^2\beta^2 \\ 1 & \beta^2 & \beta^4 \end{vmatrix}^2 = [(\alpha\beta - \alpha^2)(\beta^2 - \alpha^2)(\beta^2 - \alpha\beta)]^2.$$

Since we assume $\alpha\beta \neq 0$ and α/β is not a root of unity, we have $D_k(w_n, 0) \neq 0$, as required.

In the general case, we similarly get

$$D_k(w_n,\ 0) = \begin{vmatrix} 1 & \alpha^{k-1} & (\alpha^{k-1})^2 & \cdots & (\alpha^{k-1})^{k-1} \\ 1 & \alpha^{k-2}\beta & (\alpha^{k-2}\beta)^2 & \cdots & (\alpha^{k-2}\beta)^{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \beta^{k-1} & (\beta^{k-1})^2 & \cdots & (\beta^{k-1})^{k-1} \end{vmatrix}^2 \neq 0,$$

and the proof of the minimality is completed.

As a final remark, we note the condition for α/β not to be a root of unity. When $\alpha\beta = -b \neq 0$, then $z = \alpha/\beta$ is the root of a quadratic equation

$$p(z) = z^2 + \left(\frac{a^2}{b} + 2\right)z + 1 = 0.$$

If α/β shall not be a root of unity, we must have $z \neq \pm 1$, and $p(z)$ cannot be one of the quadratic cyclotomic polynomials $z^2 + 1$, $z^2 \pm z + 1$. Hence, we must demand that

$$\frac{a^2}{b} + 2 \neq \pm 2,\ 0,\ \pm 1 \quad \text{or} \quad -a^2 \neq 0,\ b,\ 2b,\ 3b,\ 4b.$$

REFERENCES

1. E. S. Selmer. "Linear Recurrence Relations over Finite Fields." Lecture notes; Department of Mathematics, University of Bergen, Norway, 1966.
2. M. Willett. "On a Theorem of Kronecker." *The Fibonacci Quarterly* 14, no 1 (1976):27-29.

◆◇◆◇◆