# EULER'S INTEGERS

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Euler's integers are less known than the classic Eulerian numbers, though, in figurate form, they appear since antiquity.*

First, we shall look at their origin and find their general expression; then we shall establish some of their properties and give various combinatoric applications. Several results may not have been published previously.

The notation of periodic numbers and the notion of arithmetic polynomials will be useful tools.

## I. GENERAL EXPRESSION OF EULER'S INTEGERS

Consider the infinite product

$$\pi(x) = (1 - x)(1 - x^2)(1 - x^3)\ldots$$

which Euler encountered in relation to the problem of the partition of integers. For instance, he showed that the number $p(n)$ of partitions of $n$ into integers, distinct or not, is generated by the function

$$\frac{1}{\pi(n)} = 1 + \sum_{n>0} p(n)x^n.$$

If we develop $\pi(x)$ in series, we expect *a priori* to find increasing coefficients. But, surprisingly, all coefficients are +1 or -1, isolated in gaps of zero coefficients, gaps which, on the whole, increase and tend to infinity. More precisely,

$$\pi(x) = 1 - x^{a_1} - x^{a_2} + x^{a_3} + x^{a_4} - x^{a_5} - x^{a_6} + \cdots + \varepsilon_n x^{a_n} + \cdots \tag{1}$$

The coefficients are pairwise alternately -1 and +1. In order to see the behavior of the exponents, we shall examine some initial values of *Euler's integers* $a_n$:

TABLE 1

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $a_n$ | 1 | 2 | 4 | 7 | 12 | 15 | 22 | 26 | 35 | 40 | 51 | 57 | 70 | 77 | 92 | 100 |
| $\Delta_n$ | ① | 3 | ② | 5 | ③ | 7 | ④ | 9 | ⑤ | 11 | ⑥ | 13 | ⑦ | 15 | ⑧ | |

The integers $a_n$ seem to follow a complicated law, since their rate of increase oscillates. But if we form the differences $\Delta_n = a_{n+1} - a_n$, we see that they are the integers for $n$ odd, and the odd numbers (beginning with 3) for $n$ even.

Now, we shall try to express the general term $\varepsilon_n x^{a_n}$ of the series (1) in a simple form.

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*The two kinds of Eulerian numbers $E_n$ and $A(n, k)$ are defined by:

$$\frac{1}{\cosh x} = \sum_{n \geqslant 0} E_n \frac{x^n}{n!} \quad \text{and} \quad x^n = \sum_{1 \le k \le n} A(n, k)\binom{x + k - 1}{n}.$$

Definitions.

(1) A *periodic number* $u_n = [u_1, u_2, \ldots, u_k]$ is equal to the $u_i$ in the brackets, such that $i \equiv n$, modulo $k$. So we represent a series of period $k$ by its $k$ first terms. For instance, $u_n = [a, b]$ equals $a$ or $b$, according to whether $n$ is odd or even, and $u_n = [4, -1, 0]$ is the $n^{\text{th}}$ term of the series 4, -1, 0, 4, -1, 0, 4, -1, 0, ...

(2) An *arithmetic polynomial* $P(n)$ is defined only for positive integers and takes only integer values. Contrary to an ordinary polynomial, some of its coefficients are periodic numbers. Example: $3n^2 - [4, -1, 0]n + [5, 7]$.

We shall admit the following theorem, easy to establish [1].

Theorem 1

For $u_i = [a, b], \sum_{i=1}^{n} u_i = \dfrac{(a+b)n + [a-b, 0]}{2}$;

for $u_i = [a, b]i, \sum_{i=1}^{n} u_i = \dfrac{(a+b)n^2 + [2a, 2b]n + [a-b, 0]}{4}$.

Clearly,

$$\Delta_i = \frac{i+1}{2}[1, 2] = \frac{1}{2}[1, 2]i + \frac{1}{2}[1, 2].$$

So we can calculate

$$a_n = 1 + \sum_{i=1}^{n-1} \Delta_i$$

by Theorem 1. Paying attention in brackets to the difference in parity of $n-1$ and $n$, we find

$$a_n = 1 + \frac{1}{2}\cdot\frac{3(n-1) + [0, 1]}{2} + \frac{1}{2}\cdot\frac{3(n-1)^2 + [4, 2](n-1) + [0, -1]}{4},$$

and, after simplification:

Theorem 2

The $n^{\text{th}}$ Eulerian integer is the arithmetic trinomial

$$a_n = \frac{3n^2 + [4, 2]n + [1, 0]}{8} = \left\| \frac{n}{8}(3n + [4, 2]) \right\|, \qquad (2)$$

where the double bars indicate the nearest integer.

Corollary

The general term of the series $\pi(x)$ is

$$\varepsilon_n x^{a_n} = [-1, -1, 1, 1]x^{\frac{1}{8}(3n^2 + [4, 2]n + [1, 0])}.$$

We define Euler's integers $a_n$ by Table 1, indefinitely extended by means of the two arithmetic progressions mixed in $\Delta_n$, and then deduce (2). But we have admitted (1) without proof, and so did Euler for ten years. In an article entitled "Discovery of a Most Extraordinary Law of Numbers in Relation to the Sum of Their Divisors," he said:

> I have now multiplied many factors, and I have found this progression. . . . One may attempt this multiplication and continue it as far as one wishes, in order to be convinced of the truth of this series. . . . A long time I vainly searched for a rigorous demonstration . . . and I proposed this research to some of my friends, whose competence in such questions I know; they all agreed with me on the truth of this conversion, but could not discover any source of demonstration. So it will be a known, but not yet proven truth.

Nevertheless, he finally proved it in a letter to Goldbach (1750). In the next century, various demonstrations were found, especially by Legendre [2], Cauchy, Jacobi, and Sylvester.

## II. PROPERTIES OF EULER'S INTEGERS

First a quite simple question: What is the parity of the $n^{\text{th}}$ Eulerian integer? If one observes Table 1, it seems that the same parities reappear with period 8. That is true, for

$$a_{n+8} = \frac{3(n + 8)^2 + [4, 2](n + 8) + [1, 0]}{8} = a_n + 6n + [28, 26]$$

whether $6n + [28, 26]$ is even. Likewise, we find:

Theorem 3

Modulo $k$, the Eulerian integer $a_n \equiv a_{n+4k}$ or $a_n \equiv a_{n+2k}$, according to whether $k$ is even or odd. Particularly,

$$a_n \equiv [1, 0, 1, 1, 0, 1, 0, 0], \text{ mod } 2;$$

$$a_n \equiv [1, -1, -1, 1, 0, 0], \text{ mod } 3.$$

Now a more important question: Find a characteristic property of the integers $a_n$. An integer $N$ is Eulerian, if the equation in $n$,

$$N = \frac{9n^2 + [4, 2]n + [1, 0]}{8} \quad \text{or} \quad 3n^2 + [4, 2]n + [1, 0] - 8N = 0,$$

has an integer and positive root. Therefore, its discriminator

$$[2, 1]^2 - 3[1, 0] + 24N = [4, 1] - [3, 0] + 24N = 24N + 1$$

must be a square. Conversely, if

$$24N + 1 = k^2,$$

$k$ has the form $3n + 2$ or $3n + 1$. But Eq. (2) gives

$$24a_n + 1 = 9n^2 + 3[4, 2]n + [4, 1] = (3n + [2, 1])^2.$$

So $N$ is the $n^{\text{th}}$ Eulerian integer.

Theorem 4

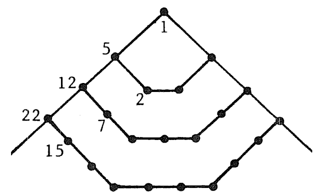
An integer $N$ is Eulerian iff $24N + 1$ is a square $k^2$. Then its rank is $\left[\frac{k}{3}\right]$ (the greatest integer $\leqslant k/3$).

The integers $a_n$ have a second characteristic property, an arithmogeometric one. If in (2) we distinguish $n$ odd and $n$ even, we have:

a) $n = 2k - 1$: $a_n = \dfrac{3k^2 - k}{3} = 1, 5, 12, 22, \ldots;$

b) $n = 2k$: $a_n = \dfrac{3k^2 + k}{2} = 2, 7, 15, 26, \ldots$

The integers $\dfrac{3k^2 - k}{2}$ are the pentagonal numbers, known since antiquity, and they count the dots of the *closed pentagons* below.




The integers $\dfrac{3k^2 + k}{2}$ also have a simple figurative signification: they count the dots of the *open pentagons*. Therefore, we call them *second-class pentagonal numbers*. Note that we also get them by $\dfrac{3k^2 - k}{2}$ for $k = -1, -2, -3, \ldots$ .

Theorem 5

Eulerian integers and pentagonal numbers are identical.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | (1) | 2 | (5) | 7 | (12) | 15 | (22) | 26 | (35) | 40 | (51) | 57 | (70) | 77 | (92) | 100 ... |

Do the integers $a_n$ satisfy a recurrence relation? Yes, for $a_n$ is an arithmetic polynomial. We know [1] that such a polynomial $a_n$ of characteristics $(d, g, p)$ (we shall define this notion directly) verifies the linear recurrence relation

$$\{(1 - a)^{d-g}(1 - a^p)^{g+1}\} = 0,$$

the exterior braces meaning that in the developed polynomial each power $a^i$ will be replaced by $a_{n-i}$. For our trinomial $a_n$ of (2), the *degree* $d = 2$, the *grade* $g = 1$ (i.e., that $n^1$ is the highest power with periodic coefficient) and the *pseudoperiod* $p = 2$ (the least common multiple of the periods of the coefficients). So

$$\{(1 - a)(1 - a^2)^2\} = \{1 - a - 2a^2 + 2a^3 + a^4 - a^5\} = 0.$$

Theorem 6

The Eulerian integers verify the recurrence relation.

$$a_n - a_{n-1} - 2a_{n-2} + 2a_{n-3} + a_{n-4} - a_{n-5} = 0.$$

We know [1] that every arithmetic polynomial $a_n$ whose recurrence relation is $\{F(a)\} = 0$ is generated by a rational fraction $f(x)/F(x)$, where $f(x)$ is of lower degree than $F(x)$. So the Eulerian integer $a_n$ is generated by a fraction

$$\frac{f(x)}{(1 - x)(1 - x^2)^2} = 1 + x + 2x^2 + 5x^3 + 7x^4 + \cdots + a_n x^n + \cdots,$$

where $f(x)$ is of degree 4 at most. Hence,

$$f(x) = 1 - x^2 + 3x^3 + x^4.$$

Theorem 7

Euler's integers are generated by the fraction

$$\frac{1 - x^2 + 3x^3 + x^4}{(1 - x)(1 - x^2)^2} = 1 + \sum_{n>0} a_n x^n.$$

As application, we now shall see Euler's integers in relation to the Eulerian function $\sigma_n$ and the partitions.

## III. EULER'S FUNCTION $\sigma(n)$

As usual, $\sigma(n)$ indicates the sum of the divisors of the integer $n$. Hence, $\sigma(8) = 1 + 2 + 4 + 8 = 15$, and $\sigma(n) = 1 + n$, iff $n$ is prime. Descartes already noted that $\sigma(nm) = \sigma(n)\sigma(m)$, iff $n$ and $m$ are relatively prime. The first values of $\sigma(n)$ are:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| $\sigma(n)$ | 1 | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 | 14 | 24 | 24 | 31 |

With respect to this table, far prolonged, Euler observed: "The irregularity of the series of the prime numbers is here intermingled. . . . It seems even that this progression is much more whimsical." Indeed the values of $\sigma(n)$ present an infinity of irregular oscillations. But Euler discovered an unexpected law in their capricious succession.

Theorem 8

The function $\sigma(n)$ verifies the recursive relation

$$\sigma(n) = \sigma(n - a_1) + \sigma(n - a_2) - \sigma(n - a_3) - \sigma(n - a_4) + \cdots \qquad (3)$$

with the convention

$$\sigma(k) = \begin{cases} n & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The $a_i$ are Euler's integers and the signs alternate pairwise.

Example: $\sigma(7) = \sigma(6) + \sigma(5) - \sigma(2) - \sigma(0) = 12 + 6 - 3 - 7 = 8.$

Admire the master's ingenious demonstration:

Take the logarithmic derivative of the two members of (1) and multiply them by $(-x)$:

$$y = \frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots = \frac{a_1 x^{a_1} + a_2 x^{a_2} - a_3 x^{a_3} - a_4 x^{a_4} + \cdots}{\pi(x)} = \frac{f(x)}{\pi(x)}.$$

Develop in series the fractions of the first member:

$$\begin{array}{rcccccccccc}
y = & x & + x^2 & + x^3 & + x^4 & + x^5 & + x^6 & + x^7 & + x^8 & + \cdots \\
 & & + 2x^2 & & + 2x^4 & & + 2x^6 & & + 2x^8 & + \cdots \\
 & & & + 3x^3 & & & + 3x^6 & & & + \cdots \\
 & & & & + 4x^4 & & & & + 4x^8 & + \cdots \\
 & & & & & + 5x^5 & & & & + \cdots \\
 & & & & & & + 6x^6 & & & + \cdots \\
 & & & & & & & + 7x^7 & & + \cdots \\
 & & & & & & & & + 8x^8 & + \cdots \\
 & & & & & & & & & - - -
\end{array}$$

Hence,

$$y = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \cdots .$$

The identity $0 \equiv -f(x) + y\pi(x)$ then gives:

$$\begin{aligned} 0 \equiv -x \quad - \quad 2x^2 \quad\quad\quad\quad\quad + \quad 5x^5 \quad\quad\quad + \quad 7x^7 + \cdots \\ +\sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \sigma(5)x^5 + \sigma(6)x^6 + \sigma(7)x^7 + \cdots \\ -\sigma(1)x^2 - \sigma(2)x^3 - \sigma(3)x^4 - \sigma(4)x^5 - \sigma(5)x^6 - \sigma(6)x^7 + \cdots \\ -\sigma(1)x^3 - \sigma(2)x^4 - \sigma(3)x^5 - \sigma(4)x^6 - \sigma(5)x^7 + \cdots \\ +\sigma(1)x^6 + \sigma(2)x^7 + \cdots \end{aligned}$$

Relation (3) states that the coefficient of $x^n$ in the second member of the preceding identity if zero. We see it clearly when we look at the coefficient of $x^7$ for example.

## The Series $u = \frac{\sigma(n)}{n}$

We proved that *the function* $\frac{\sigma(k!)}{k!}$ *increases and we shall see that it tends to infinity with* $k$.

Let $P_1, P_2, \ldots, P_r$ be the prime numbers up to $P_r$. Then

$$\frac{\sigma(P_i)}{P_i} = 1 + \frac{1}{P_i} \quad \text{and} \quad \frac{\sigma(P_r!)}{P_r!} > \left(1 + \frac{1}{P_1}\right)\left(1 + \frac{1}{P_2}\right) \cdots \left(1 + \frac{1}{P_r}\right).$$

Hence,

$$L\frac{(P_r!)}{P_r!} > L\left(1 + \frac{1}{P_1}\right) + L\left(1 + \frac{1}{P_2}\right) + \cdots + L\left(1 + \frac{1}{P_r}\right) = \sum\frac{1}{P_i} - \frac{1}{2}\sum\frac{1}{P_i^2} + \frac{1}{3}\sum\frac{1}{P_i^3} \cdots,$$

the sums being taken from $i = 1$ to $i = r$. We know that $\sum\frac{1}{P_i} \to \infty$ with $r$, while the other sums converge. Therefore, $\frac{\sigma(P!)}{P!} \to \infty$ with $P$, and also $\frac{\sigma(k!)}{k!} \to \infty$ with $k$.

What a curious series is $u_n = \frac{\sigma(n)}{n}$ ! Obviously, $u_n \geqslant 1$. It oscillates irregularly—probably between 1 and 6 for $n < 10^{17}$—but it presents an initial regularity: it has a relative extremum for each $n < 62$. The extreme example is likely

$$n = 2^{12} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \simeq 0.998 \times 10^{17}$$

with $u_n \simeq 5999$. It contains at once a decreasing series $u(P_i)$, which tends to 1, a constant series $u(E_k) = 2$, where $E_k$ is the $k^{\text{th}}$ Euclidean integer, increasing series $u(a^k)$, which tend to finite numbers if $k \to \infty$, and an increasing series $u(k!)$, which tends to infinity. Furthermore, $u_{nm} = u_n u_m$ if $n$ and $m$ are relatively prime, and $u_{nm} < u_n u_m$ if not. For a prime $P$ and an arbitrary integer $k$, $u(P^k) < 2$. While

$$u_1 = 1,\ u_6 = 2,\ u_{120} = 3,\ u_{30240} = 4,$$

the least known $n$ for $u_n = 6$ exceeds $10^{28}$ and the least $n$ for $u_n = 8$ is gigantic [3]. Descartes, Fermat, and others, assiduously searched for values of $n$ for which $u_n$ is an integer. All the found values, save 1 and 6, are multiples of 4.

Perfect numbers can be defined by $u_n = 2$. Euler proved that *the only even perfect numbers are the Euclidean integers* $p(p+1)/2$, where $p = 2^{k+1} - 1$ is prime. *Can an odd perfect number exist*? Nobody knows. But we know that the order of such an odd $n$ would be at least $10^{200}$ [3]. The difficulty of this millenary question has been compared to that of the transcendency of $\pi$ (previously, to Lindemann's historical demonstration) or that of Fermat's open problem. More generally:

Conjecture

For an odd $n$, save 1, the number $u_n$ is never an integer.

Here are some initial values of $u(k!)$, approached for $k > 5$:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 --- | 13 --- | 20 --- | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u(k!)$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.36 | 3.84 | 3.95 | 4.08 | 4.22 | --4.99 | --5.52 | --5.95 |

Generally, $u_n < u(k!)$ for $n < k!$. But never: $30240 < 8!$, although $u(30240) = 4$ (found by Descartes) exceeds $u(8!) \simeq 3.95$.

## IV. PARTITIONS INTO DISTINCT OR UNRESTRICTED PARTS

Another Eulerian formula is strangely similar to (3). It concerns the number $p(n)$ of partitions of $n$, into integers distinct or not, whose first values are:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 | 56 | 77 | 101 | 135 | 176 | 231 |

Theorem 9

The number of unrestricted partitions of verifies the recursive relation

$$p(n) = p(n - a_1) + p(n - a_2) - p(n - a_3) - p(n - a_4) + \cdots$$

with the convention

$$p(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The $a_i$ are Euler's integers and the signs alternate pairwise.

This formula results directly from the fact, mentioned at the beginning, that $p(n)$ is generated by $1/\pi(x)$.

Is it not fabulous that two beings, so disparate as $\sigma(n)$ and $p(n)$ (sum of the divisors of $n$ and number of its partitions) follow the same recursive law (aside from a slight detail: $\sigma_0 = n$, $p_0 = 1$)?

Could a similar recursive law exist, perhaps not linear, for the prime numbers $P_n$?

Recently D. R. Hickerson found an interesting relation between the numbers of distinct or unrestricted partitions [4]:

Theorem 10

The number $p_n$ of unrestricted partitions of $n$ and the number $q_n$ of its partitions into distinct parts are related by

$$q_n = p_n - p_{n-2a_1} - p_{n-2a_2} + p_{n-2a_3} + p_{n-2a_4} - \cdots$$

with the convention

$$p = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The signs alternate pairwise.

Starting from the generating functions of $p_n$ and $\sigma_n$, we have established an unexpected relation between them:

Theorem 11

The arithmetic functions $p_n$ and $\sigma_n$ are related by

$$\sigma_n = a_1 p_{n-a_1} + a_2 p_{n-a_2} - a_3 p_{n-a_3} - a_4 p_{n-a_4} + \cdots$$

with the convention

$$p_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0. \end{cases}$$

The signs alternate pairwise.

Conjecture

For $n > 6$, the function $Lp_n/\sqrt{n}$ increases and $Lp_n/n$ decreases. But

$$\frac{Lp_{20}}{\sqrt{20}} > 1.44 \quad \text{and} \quad \frac{Lp_{20}}{20} < 0.33.$$

Hence,

$$e^{1.44\sqrt{n}} < p_n < e^{0.33n} \text{ for } n > 20. \tag{4}$$

Remarks

1) An asymptotical value for $p_n$ was found by Hardy and Ramanujan:

$$p_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}n} \sim \frac{e^{2.57\sqrt{n}}}{6.93n}.$$

Consequently (4) *is proved for n great.*

2) We know that *the number of partitions of n into unrestricted parts is* $2^{n-1}$, *if the order of the summands is relevant.*

Example: For $n = 3 = 1 + 2 = 2 + 1 = 1 + 1 + 1$, this number is $2^2$.

Therefore, $$p_n < 2^{n-1} \text{ for } n > 2.$$

Theorem 12

Let $q''_n$ and $q'_n$, respectively, be the numbers of partitions of $n$ into an even or an odd number of distinct parts. If the integer $n$ is not Eulerian, $q'_n = q''_n$; for a Eulerian integer $a_n$, $q''_{a_n} = q'_{a_n} + [-1, -1, 1, 1]$, the periodic number being related to the rank $n$ of $a_n$.

Corollary

The number of partitions of an integer $n$ into distinct parts is odd iff is Eulerian. Euler stated that this number equals the number of partitions of $n$ in which all parts, distinct or not, are odd.

The coefficient of $x^N$ is the same in the series

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 + c_1x + c_2x^2 + \cdots + c_Nx^N + \cdots$$

and in the polynomial

$$(1-x)(1-x^2)(1-x^3)\cdots(1-x^N) = 1 + c_1x + c_2x^2 + \cdots + c_Nx^N + x^{N+1}P(x). \tag{5}$$

By developing the product (5) without reducing similar terms, we get, with coefficient (+1), every $x^N$ whose exponent appears as a partition of $N$ in an even number of distinct terms, and with coefficient (-1), each $x^N$ whose exponent appears as a partition of $N$ in an odd number of distinct integers. Therefore,

$$c_N = q''_N - q'_N.$$

But in (1), $\varepsilon_n = [-1, -1, 1, 1]$ or 0, according to whether $N$ is Eulerian or not.

## Remarks

1) Although Theorem 12 follows easily from identity (1), Legendre seems to have been the first to state it [2].

2) Now the great gaps in the series (1) are explained: they simply signify that generally an integer has as many partitions in an even as in an odd number of distinct parts.

3) An odd $q_n$ is characteristic of Eulerian integers, *as an odd* $\sigma_n$ *is characteristic of squares or double squares*. But the problem of the parity of $p_n$ is still open.

## V. PARTITIONS INTO PARTS OF GIVEN VALUES

The following text of Euler shows with charming simplicity his enthusiasm for his amazing formula (3). His integers seem to be still a little mysterious to him.

> We are the more surprised by this beautiful property, as we see no relation between the composition of our formula and the divisors whose sums concern the proposition. The progression of the numbers 1, 2, 5, 7, 12, 15, ... not only seems to have no relation to the subject, but—as the law of their numbers is interrupted and they are a mixture of two different progressions: 1, 5, 12, 22, 35, 51, ... and 2, 7, 15, 26, 40, 57, ...—it almost seems that such an irregularity could not exist in analysis.

So Euler was surprised that $a_n$ takes its values from two progressions, trinomials of the second degree. However, notwithstanding what he believed, one often meets in analysis series of integers that take their values from several polynomials: *the arithmetic polynomials*, which all have a generating rational fraction and satisfy a linear recurrence relation. It is piquant to see that such series occur, particularly in a question which Euler examined at length: the partition into parts of given values [5].

## Example 1

In how many ways can $n$ identical objects be divided in groups of 12, 13, and 17 pieces?

This is equivalent to finding the number $j_n$ of nonnegative integer solutions of the equation

$$12x + 13y + 17z = n. \tag{6}$$

Those problems are solved by a general theorem, whose first part is due to Euler:

Theorem 13

The number $j_n$ of nonnegative solutions of the diophantine equation

$$\sum_{i=1}^{r} \alpha_i x^i = n$$

with positive coefficients, is generated by the fraction

$$\frac{1}{(1 - t^{\alpha_1})(1 - t^{\alpha_2}) \ \ldots \ (1 - t^{\alpha_r})} = \sum_{n \geqslant 0} j_n t^n.$$

The function $j(n)$ is an arithmetic polynomial, whose pseudoperiod is the least common multiple of the $\alpha_i$, its degree $r - 1$ and its grade $m - 1$, $m$ being the greatest number of coefficients $\alpha_i$ that have a common divisor other than 1 [1].

So, for Eq. (6), $j_n$ is an arithmetic trinomial whose characteristics are (2, 0, 2652). More precisely, we know [1] that $j_n$ verifies a relation of the form

$$2(12 \times 13 \times 17)j_n = n^2 + (12 + 13 + 17)n + u_n,$$

where $u_n$ is a number of period $12 \times 13 \times 17 = 2652$.

You may think the 2652 components of the periodic number $u_n$ long to calculate, and the expression of $j_n$ long to write. Not at all. The calculation of $u_n$ is performed in an instant by the computer (with the program for the resolution of a system of linear equations, which every computing center has) and

$$j_n = \left\| \frac{n^2 + 42n + 100(A_n - B_n)}{5304} \right\|,$$

where the periodic numbers

$$A_n = [5, 21, 25, 17, -2, 17, 25, 21, 5, 30, 42, 42, 30]$$

and

$$B_n = [-2, 17, 6, 17, -2, 0, 24, 17, 33, 17, 24, 0]$$

have, respectively, 13 and 12 components.

*The error is at most* 1, *for*

$$j_n \simeq \left\| \frac{n(n + 42)}{5304} \right\|.$$

Example 2

What is the number of solutions in nonnegative integers of the equation

$$x + 2y + 6z + 3 = 3n?$$

We have shown that *this number is Euler's integer* $a_n$.

Note that $a_n$ has the characteristics (2, 1, 2), while, for the diophantine equation

$$x + 2y + 6z = n$$

the number of nonnegative solutions is, by Theorem 13, an arithmetic trinomial of characteristics (2, 1, 6).

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