ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS and SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-526 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Find all ordered pairs (m, n) of positive integers for which there is an integer x satisfying the equation

$$F_m F_n x^2 - [F_m(F_m,\ F_n) + F_n F_{(m,n)}]x + (F_m,\ F_n)F_{(m,n)} = 0.$$

Here (r, s) denotes the greatest common divisor of r and s.

B-527 *Proposed by L. Cseh and I. Merenyi, Cluj, Romania*

Do as in B-526 with the equation replaced by

$$(F_m,\ F_n)x^2 - (F_m + F_n)x + F_{(m,n)} = 0.$$

B-528 *Proposed by Herta T. Freitag, Roanoke, VA*

For nonnegative integers n, prove that

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_{i+1}^2 = 5^n F_{2n+3}.$$

B-529 *Proposed by Herta T. Freitag, Roanoke, VA*

For positive integers n, find a compact form for $\sum_{i=0}^{2n} \binom{2n}{i} F_{i+1}^2$.

B-530 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $\alpha = (1 + \sqrt{5})/2$. For n an odd positive integer, prove that the continued fraction

$$L_n + \cfrac{1}{L_n + \cfrac{1}{L_n + \cdots}} = \alpha^n.$$

B-531 *Proposed by Michael Eisenstein, San Antonio, TX*

For n an even positive integer, prove that

$$L_n - \cfrac{1}{L_n - \cfrac{1}{L_n - \cdots}} = \alpha^n.$$

SOLUTIONS

Even Sum of Fibonacci Products

B-502 *Proposed by Herta T. Freitag, Roanoke, VA*

Given that h and k are integers with $h+k$ an integral multiple of 3, prove that $F_k F_{k-h-1} + F_{k+1} F_{k-h}$ is even.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Letting $t = n + 1$ in (I_{26})—see p. 59 of Verner E. Hoggatt, Jr., *Fibonacci and Lucas Numbers* (Boston: Houghton Mifflin Co., 1969)—yields the following identity:

$$F_{m+t} = F_{m+1} F_t + F_m F_{t-1}. \qquad (*)$$

Thus,

$$\begin{aligned} F_k F_{k-h-1} + F_{k+1} F_{k-h} &= F_{k+1} F_{k-h} + F_k F_{k-h-1} \\ &= F_{k+(k-h)} \ [\text{by } (*)] \\ &= F_{2k-h} \\ &= F_{3k-(h+k)}. \end{aligned}$$

Because $h + k$ is a multiple of 3, 3 divides $3k - (h + k)$, hence $2 = F_3$ divides $F_{3k-(h+k)}$.

Also solved by Wray G. Brady, Paul S. Bruckman, L. Cseh, M. J. DeLeon, C. Georghiou, Walther Janous, L. Kuipers, Graham Lord, I. Merenyi, Bob Prielipp, Heinz-Jürgen Seiffert, Sahib Singh, and the proposer.

Even Perfect Numbers Mod 7

B-503 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Prove that every even perfect number except 28 is congruent to 1 or −1 modulo 7.

Solution by L. Cseh, Cluj, Romania

It is well known that every even perfect number is of the form

$$2^{p-1}(2^p - 1),$$

where p is prime and so is $(2^p - 1)$. Every prime, except 3 is of the form

$$3k + 1 \quad \text{or} \quad 3k + 2.$$

Thus, we have

$$2^{3k}(2^{3k+1} - 1) \equiv 1 \cdot (1 \cdot 2 - 1) \equiv 1 \pmod 7$$
$$2^{3k+1}(2^{3k+2} - 1) \equiv 2 \cdot (4 - 1) \equiv 6 \equiv -1 \pmod 7,$$

and because for $p = 3$ we obtain 28, the proof is complete.

Also solved by Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, C. Georghiou, Walther Janous, H. Klauser and M. Wachtel, L. Kuipers, Graham Lord, I. Merenyi, Bob Prielipp, Sahib Singh, and the proposer.

Triangular Fibonacci Numbers Mod 24

B-504 *Proposed by Charles R. Wall*

Prove that if n is an odd integer and F_n is in the set $\{0, 1, 3, 6, 10, \ldots\}$ of triangular numbers, then $n \equiv \pm 1 \pmod{24}$.

Solution by Leonard Dresel, University of Reading, England

If F_n is in the set of triangular numbers, then there is an integer k such that $F_n = \frac{1}{2}k(k + 1)$, so that $8F_n + 1 = (2k + 1)^2$ is a perfect square. Reducing this modulo 9, we have

$$8F_n + 1 \text{ is a quadratic residue modulo } 9.$$

The Fibonacci sequence reduced modulo 9 is periodic with period 24, and for the odd integers n, we have

$$\begin{array}{rcccccccccccccl} n & \equiv 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & \pmod{24} \\ F_n & \equiv 1 & 2 & 5 & 4 & 7 & 8 & 8 & 7 & 4 & 5 & 2 & 1 & \pmod 9 \\ 8F_n + 1 & \equiv 0 & 8 & 5 & 6 & 3 & 2 & 2 & 3 & 6 & 5 & 8 & 0 & \pmod 9. \end{array}$$

By squaring the numbers 0, 1, 2, 3, and 4, we find that the quadratic residues modulo 9 are 0, 1, 4, 7. Hence, the only quadratic residue in the sequence for $8F_n + 1 \pmod 9$ is the number 0, and this occurs only for $n \equiv \pm 1 \pmod{24}$.

We can extend this result in various ways. For example, by reducing the sequence $8F_n + 1$ modulo 11, we obtain the further condition $n \equiv \pm 1 \pmod{10}$.

Also solved by Paul S. Bruckman and the proposer.

Sum of Lucas Products

B-505 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$N = N(m, a) = L_{m-2a}L_m - L_{m+1-2a}L_{m-1},$$

where m and a are positive integers. Prove or disprove that N is: (a) always (exactly) divisible by 5; (b) never divisible by 3, 4, 6, 7, 8, 9, or 11; and (c) divisible by 10 if $a \equiv 2 \pmod 3$.

Solution by C. Georghiou, University of Patras, Greece

When L_n is replaced by $\alpha^n + \beta^n$, we get

$$N = L_{m-2a}L_m - L_{m+1-2a}L_{m-1} = (-1)^m(L_{2a} + L_{2a-2}) = (-1)^m 5F_{2a-1};$$

therefore, N is divisible by 5.

Next, we take the following properties of the Fibonacci numbers as known (otherwise, they can easily be established):

$$F_n \equiv 0 \pmod 3 \quad \text{iff} \quad n \equiv 0 \pmod 4 \tag{1}$$

$$F_n \equiv 0 \pmod 4 \quad \text{iff} \quad n \equiv 0 \pmod 6 \tag{2}$$

$$F_n \equiv 0 \pmod 7 \quad \text{iff} \quad n \equiv 0 \pmod 8 \tag{3}$$

$$F_n \equiv 0 \pmod{11} \quad \text{iff} \quad n \equiv 0 \pmod{10} \tag{4}$$

$$F_n \equiv 0 \pmod 2 \quad \text{iff} \quad n \equiv 0 \pmod 3 \tag{5}$$

Now (1) $\Rightarrow N \not\equiv 0$ (mod 3 or mod 6 or mod 9),
(2) $\Rightarrow N \not\equiv 0$ (mod 4 or mod 8),
(3) $\Rightarrow N \not\equiv 0 \pmod 7$,
(4) $\Rightarrow N \not\equiv 0 \pmod{11}$, and finally,
(5) $\Rightarrow N \equiv 0 \pmod{10}$ iff $2a - 1 \equiv 0 \pmod 3$ or $a \equiv 2 \pmod 3$.

Also solved by Paul S. Bruckman, L. Cseh, M. J. DeLeon, Walther Janous, L. Kuipers, Graham Lord, Bob Prielipp, Sahib Singh, and the proposer.

Fibonacci and Lucas Convolutions

B-506 *Proposed by Heinz-Jürgen Sieffert, student, Berlin, Germany*

Let $G_n = (n + 1)F_n$ and $H_n = (n + 1)L_n$. Prove that:

(a) $$\sum_{k=0}^{n} G_k G_{n-k} = \frac{(n+2)(n+3)}{30}H_n - \frac{2}{25}H_{n+2} + \frac{4}{25}F_{n+3};$$

(b) $$\sum_{k=0}^{n} H_k H_{n-k} = \frac{(n+2)(n+3)}{6}H_n + \frac{2}{5}H_{n+2} - \frac{4}{5}F_{n+3}.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

Let

$$U(x) = x/(1 - x - x^2) = \frac{1}{\sqrt{5}}((1 - \alpha x)^{-1} - (1 - \beta x)^{-1}) = \sum_{n=0}^{\infty} F_n x^n; \tag{1}$$

$$V(x) = (2 - x)/(1 - x - x^2) = P + Q = \sum_{n=0}^{\infty} L_n x^n,$$

where

$$P = (1 - \alpha x)^{-1} \quad \text{and} \quad Q = (1 - \beta x)^{-1}.$$

Also, let

$$A(x) = (xU(x))', \ B(x) = (xV(x))'. \tag{2}$$

Then

$$A(x) = \sum_{n=0}^{\infty} G_n x^n, \ B(x) = \sum_{n=0}^{\infty} H_n x^n. \tag{3}$$

Note that $(xP)' = P^2$, $(xQ)' = Q^2$. Hence,

$$A(x) = 5^{-1/2}(P^2 - Q^2),\ B(x) = P^2 + Q^2. \tag{4}$$

Let

$$R(x) = P^4 + Q^4,\ S(x) = P^2Q^2. \tag{5}$$

Then

$$R(x) = \sum_{n=0}^{\infty}\binom{n+3}{3}(\alpha^n + \beta^n)x^n,$$

or

$$R(x) = \frac{1}{6}\sum_{n=0}^{\infty}(n+2)(n+3)H_n x^n. \tag{6}$$

Also,

$$S(x) = (1 - x - x^2)^{-2} = \frac{1}{4}(U(x) + V(x))^2$$
$$= \frac{1}{4}\{(1 + 5^{-1/2})P + (1 - 5^{-1/2})Q\}^2 = \frac{1}{5}\{\alpha^2P^2 + 2PQ + \beta^2Q^2\};$$

hence,

$$5S(x) = \alpha^2P^2 + U(x) + V(x) + \beta^2Q^2$$
$$= \sum_{n=0}^{\infty}(n+1)L_{n+2}x^n + \frac{2}{\sqrt{5}}(\alpha P - \beta Q)$$
$$= \sum_{n=0}^{\infty}\{(n+1)L_{n+2} + 2F_{n+1}\}x^n$$
$$= \sum_{n=0}^{\infty}\{(n+3)L_{n+2} + 2(F_{n+1} - L_{n+2})\}x^n,$$

or

$$S(x) = \frac{1}{5}\sum_{n=0}^{\infty}(H_{n+2} - 2F_{n+3})x^n. \tag{7}$$

Now,

$$(A(x))^2 = \left(\sum_{n=0}^{\infty}G_n x^n\right)^2 = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}G_kG_{n-k};$$

also, however, from (4) and (5), $5(A(x))^2 = R(x) - 2S(x)$. Using (6) and (7):

$$\sum_{k=0}^{n}G_kG_{n-k} = \frac{1}{30}(n+2)(n+3)H_n - \frac{2}{25}(H_{n+2} - 2F_{n+3}),$$

or

$$\sum_{k=0}^{n}G_kG_{n-k} = \frac{1}{30}(n+2)(n+3)H_n - \frac{2}{25}H_{n+2} + \frac{4}{25}F_{n+3}. \tag{8}$$

Likewise,

$$(B(x))^2 = \left(\sum_{n=0}^{\infty}H_n x^n\right)^2 = \sum_{n=0}^{\infty}x^n\sum_{k=0}^{n}H_kH_{n-k} = R(x) + 2S(x)$$
$$= \frac{1}{6}\sum_{n=0}^{\infty}(n+2)(n+3)H_n x^n + \frac{2}{5}\sum_{n=0}^{\infty}(H_{n+2} - 2F_{n+3})x^n,$$

so

$$\sum_{k=0}^{n}H_kH_{n-k} = \frac{1}{6}(n+2)(n+3)H_n + \frac{2}{5}H_{n+2} - \frac{4}{5}F_{n+3}. \tag{9}$$

Also solved by C. Georghiou, L. Kuipers, J. Suck, Gregory Wulczyn, and the proposer.

Mixed Convolution

B-507 *Proposed by Heinz-Jürgen Sieffert, Berlin, Germany*

Let G_n and H_n be as in B-506. Find a formula for $\sum_{k=0}^{n} G_k H_{n-k}$ similar to the formulas in B-506.

Solution by Paul S. Bruckman, Fair Oaks, CA

We follow the notation introduced in the solution to B-506, and note that

$$A(x)B(x) = \sum_{n=0}^{\infty} G_n x^n \cdot \sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} G_k H_{n-k}.$$

On the other hand,

$$A(x)B(x) = 5^{-1/2}(P^4 - Q^4) = 5^{-1/2} \sum_{n=0}^{\infty} \binom{n+3}{3} (\alpha^n - \beta^n) x^n$$

$$= \frac{1}{6} \sum_{n=0}^{\infty} (n+2)(n+3) G_n x^n.$$

Hence,

$$\sum_{k=0}^{n} G_k H_{n-k} = \frac{1}{6}(n+2)(n+3)G_n.$$

Also solved by C. Georghiou, L. Kuipers, J. Suck, Gregory Wulczyn, and the proposer.

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4. A. G. Shannon & A. F. Horadam. "Infinite Classes of Sequence-Generated Circles." *The Fibonacci Quarterly* (to appear).
5. L. G. Wilson. "Fibonacci Sequences." Private communication, 1982.

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