SOME IDENTITIES ARISING FROM THE FIBONACCI NUMBERS OF CERTAIN GRAPHS

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Tichy and Prodinger [5] have defined the Fibonacci number of a graph G to be the number of independent vertex sets I in G; recall that I is independent if no two of its vertices are adjacent. Following Tichy and Prodinger, we denote the Fibonacci number of G by $F(G)$. If k is a nonnegative integer, we will denote the k-element independent vertex sets in G by $F_k(G)$. It is clear that $\sum F_k(G) = F(G)$. Kreweras [4] (see also [3]) has introduced the notion of the Fibonacci polynomial,

$$F(x) = \sum_{k \geqslant 0} \binom{n-k}{k} x^k.$$

We define the more general concept of the Fibonacci polynomial of a graph G, denoted $F_G(x)$. In case G is a path on n vertices,

$$F_G(x) = \sum_{k \geqslant 0} \binom{n-k+1}{k} x^k,$$

which closely resembles Kreweras' polynomial. Before defining $F_G(x)$, we compute $F_k(P_n)$, P_n the path on n vertices, and $F_k(C_n)$, C_n the cycle on n vertices.

Proposition 1

(i) $F_0(P_n) = 1$;

(ii) $F_1(P_n) = n$;

(iii) $F_k(P_{n+1}) = F_k(P_n) + F_{k-1}(P_{n-1})$ for $1 \leqslant k \leqslant \left[\frac{n+2}{2}\right]$;

(iv) $F_k(P_n) = \binom{n-k-1}{k}$ for $0 \leqslant k \leqslant \left[\frac{n+1}{2}\right]$.

Proof: The first two statements are obvious. To verify (iii), consider those k-element independent sets that contain the initial point of the path and those that do not. Finally, (iv) may be verified using (iii) and induction on n. ∎

Proposition 1 provides a natural graph-theoretic interpretation of the well-known formula

$$\sum_{k \geqslant 0} \binom{n-k+1}{k} = F_{n+1},$$

the $n + 1^{\text{th}}$ Fibonacci number. The right side of the equality is the number of independent sets of a path with n vertices. The left side is the sum over all k of the number of k-element independent sets. The following proposition will enable us to give an analogous identity involving Lucas numbers, and a graph-theoretic interpretation of that identity.

Proposition 2

(i) $F_0(C_n) = 1$;

(ii) $F_1(C_n) = n$;

(iii) $F_k(C_n) = F_k(P_{n-1}) + F_{k-1}(P_{n-3})$ for $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$ and $n \geqslant 3$;

(iv) $F_k(C_n) = \frac{n}{k}\binom{n-k-1}{k-1}$ for $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$ and $n \geqslant 3$.

Proof: Again, the first two statements are obvious. To verify (iii), fix a vertex x in C_n. Consider those k-element independent sets that contain x and those that do not; use (iv) of Proposition 1. To verify (iv), we use (iii):

$$\begin{aligned} F_k(C_n) &= F_k(P_{n-1}) + F_{k-1}(P_{n-3}) \\ &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \\ &= \frac{n}{k}\binom{n-k-1}{k-1}. \ \blacksquare \end{aligned}$$

We now use Proposition 2 to obtain an identity analogous to that following Proposition 1. L_n denotes the n^{th} Lucas number.

Proposition 3

For $n \geqslant 3$, $1 + \sum_{k \geqslant 1} \frac{n}{k}\binom{n-k-1}{k-1} = L_n$.

Proof: The right side is the number of independent sets in C_n (see [5]). The left side is the sum over k of the number of k-element independent subsets. ■

We now pause to establish some notation and state a definition. If G and H are graphs, we will denote by $G \cdot H$ the standard composition or lexicographic product (see [1]). That is, $G \cdot H$ is the graph constructed by replacing each vertex v of G by an isomorphic copy H_v of H, and by joining each vertex of H_v to each vertex of H_w whenever v is adjacent to w in G. We define the Fibonacci polynomial of G, F_G, by $F_G(x) = F(G \cdot k_x)$ for positive integers x. As usual, k_x is the complete graph on x vertices. That F_G is a polynomial follows from the next proposition.

Proposition 4

Let G be a graph, and let $F_k = F_k(G)$ for $k \geqslant 0$. Then $F_G(x) = \sum_{k \geqslant 0} F_k x^k$.

Proof: To obtain a k-element independent set in $G \cdot k_x$, one must first choose a k-element independent set in G, and then choose one of the x vertices in each of the k chosen copies of k_x. ■

The study of the Fibonacci polynomial of G thus reduced to the study of the coefficients $F_k(G)$. For example, the constant term of $F_G(x)$ is 1, the linear term is nx, and the coefficient of x^2 is $\binom{n}{2} - m$, where m is the number of edges of G. The degree of $F_G(x)$ is the independence number of G, that is, the number of vertices in the largest independent set.

We obtain some combinatorial identities by expanding the Fibonacci polynomials of paths and cycles.

Theorem 5

Let x be a positive integer, and let n be a nonnegative integer. Let ℓ be $\frac{1}{2}(1 \pm \sqrt{1 + 4x})$. Then,

$$\sum_{k \geqslant 0} \binom{n - k + 1}{k} x^k = \frac{1}{2\ell - 1}(\ell^{n+2} - (1 - \ell)^{n+2}).$$

Proof: We compute the Fibonacci polynomial of P_n in two ways. First, use Proposition 4 and Proposition 1 to get

$$\sum_{k \geqslant 0} \binom{n - k + 1}{k} x^k.$$

As a second approach, we derive and solve a second-order linear recursion for $a_n = F(P_n \circ k_x)$. Clearly, $a_0 = 1$ and $a_1 = x + 1$. Divide the independent sets in $P_n \circ k_x$ into those that contain a vertex in the last stalk and those that do not. There are xa_{n-2} of the first type, and a_{n-1} of the second type. Hence, $a_n = a_{n-1} + xa_{n-2}$. This recursion has characteristic equation $\lambda^2 - \lambda - x = 0$. Solving this equation, subject to the initial conditions, yields

$$a_n = F(P_n \circ k_x) = \frac{1}{2\ell - 1}(\ell^{n+2} - (1 - \ell)^{n+2}). \blacksquare$$

Note that the identity in Theorem 5 is true for infinitely many values of x. Hence, it is in fact true for all complex numbers x. The same remark applies to the following theorem.

Theorem 6

Let x be a positive integer, and let n be a nonnegative integer. Let ℓ be $\frac{1}{2}(1 \pm \sqrt{1 + 4x})$. Then,

$$1 + \sum_{k \geqslant 1} \frac{n}{k} \binom{n - k - 1}{k - 1} x^k = \ell^n + (1 - \ell)^n.$$

Proof: We compute the Fibonacci polynomial of C_n in two ways. First, we use Propositions 2 and 4 to get

$$1 + \sum_{k \geqslant 1} \frac{n}{k} \binom{n - k - 1}{k - 1} x^k.$$

Now we use Theorem 5. Let S be a fixed stalk in $C_n \circ k_x$. Divide the independent sets in $C_n \circ k_x$ into those that contain a vertex in S and those that do not. There are

$$x\left(\frac{1}{2\ell - 1}\right)(\ell^{n-1} - (1 - \ell)^{n-1})$$

independent sets of the first type and

$$\frac{1}{2\ell - 1}(\ell^{n+1} - (1 - \ell)^{n+1})$$

of the second type. Adding, and substituting $x = \ell^2 - \ell$ yields the theorem. $\blacksquare$

The identity of Theorem 5 is known. See, for example, [2, p. 76]. But our approach seems to provide a new interpretation for this identity. We believe that new identities may be obtained by expanding Fibonacci polynomials of graphs.

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